An Efficient Auction for Budget Constrained Bidders with Multi-dimensional Types

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Abstract

I study an auction for a divisible good where bidders have private values and private budgets. My main result shows that when bidders have full-support beliefs over their rivals' types, a clinching auction played by proxy-bidders implements a Pareto efficient outcome. The auction is not dominant strategy implementable, but it can be solved using two rounds of iterative deletion of weakly dominated strategies. The predictions do not require that bidders share a common prior and they place no restrictions on higher-order beliefs. The results provide a contrast to recent work that shows there is no mechanism with VCG's desired incentive and efficiency properties when bidders have non-quasilinear preferences and multi-dimensional types.

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1 Introduction

The Vickrey-Clarke-Groves (VCG) mechanism is celebrated as a major achievement in the theory of mechanism design. However, the practical applicability of the VCG mechanism is limited when there is a difference between a bidder's willingness to pay for goods and her ability to pay. In particular, the VCG mechanism loses its desired incentive and efficiency properties when bidders are budget constrained. In this paper, I propose an alternative to the VCG mechanism for multi-unit auction environments where bidders have private values and private budgets. Specifically, I show when bidders have full-support beliefs of their rivals' types, a clinching auction played by proxy bidders yields a Pareto efficient outcome. The mechanism is not dominant strategy implementable, but instead it can be solved using two rounds of iterative elimination of dominated strategies.

The question of auction design with budgets is important for practical applications, as budgets feature prominently in many well-studied auction markets and directly affect bid behavior in standard auctions.[1] In online ad auctions used by Yahoo! and Google, Dobzinski, Lavi, and Nisan (2012) argue that budgets are an important consideration when firms determine their bids. In another example, Rothkopf (2007) recalls his consulting experience to argue that budgets limit the usefulness of the Vickrey auction. He advised a firm that valued a license at \$85 million, yet was only able to finance a \$65 million bid. Maskin (2000) claims that budgets should be a consideration for governments selling publicly-owned assets. He says that budgets are especially important in developing economies, as bidders are less likely to have easy access to well-functioning credit markets.

VCG is inefficient when bidders have budgets but prior work provides alternatives to VCG that are efficient, under the restriction that budgets are public (see Maskin (2000) and Dobzinski, Lavi, and Nisan (2012)). In their paper, Dobzinski, Lavi, and Nisan (2012) (hereafter, DLN) show that a clinching auction is efficient and dominant strategy incentive compatible when budgets are public. The public budget restriction is useful from a theoretical perspective, as it allows us to model bidders as having one-dimensional types. However, in practice, bidders' access to credit and liquidity is often private information. For example, Google's auctions for television ad space solicit information on bidders values and budgets (see Nisan et al. (2009)). In addition, DLN argue that budgets are more salient to bidders than their valuations.

Thus, understanding efficient auction design with private budgets is important for prac-

[1]For single object auctions, Che and Gale (1996, 1998, 2006) look at the private values case, and Fang and Parreiras (2002, 2003) and Kotowski (2018) study interdependent value cases. In the multiunit auctions literature, see Rothkopf (1977), Palfrey (1980), Benoit and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009).

tical applications. Unfortunately, DLN provides a negative result on this topic. They show that when bidders have private budgets, there is no dominant strategy incentive compatible mechanism that gives an efficient outcome and satisfies weak budget balance. More recently, Baisa (2017) shows that there is no mechanism that retains VCG's desired properties if bidders have multi-dimensional private information. While DLN show that a clinching auction is efficient when budgets are public, when budgets are private the clinching auction loses its desired incentive and efficiency properties. This is because when bidders have private budgets, they have an incentive to overreport their budgets allowing them to win more units at a lower price per unit. DLN use this result to motivate their impossibility theorem. In a follow up paper, Bhattacharya et al. (2010) recognize the inefficiencies that result from bidders overreporting budgets, and show that a clinching auction is efficient when bidders are exogenously assumed unable to overreport their private budgets. This assumption simplifies the design problem, but it also limits the applicability of their results. In practice, we assume the auctioneer is unable to observe bidders' private information. If a bidder is able to overreport her private information, then it is still the case that DLN's impossibility result implies that there is no efficient auction that is dominant strategy implementable.[2]

In contrast to this prior work, which gives impossibility results related to efficient auction design in multi-dimensional type spaces without quasilinearity, I study the efficient auction design problem and obtain positive results. I obtain a positive result by relaxing using a slightly weaker solution concept than dominant strategy. In particular, I show that a clinching auction played by proxy bidders is efficient and solvable using two rounds of iterative deletion of dominated strategies. The mechanism satisfies ex-post individual rationality and provides bidders with an incentive to report their private information truthfully.

There are two main reasons for these contrasting results. First, prior work studying auctions with private budgets place no restrictions on bidders' beliefs over their rivals' types. I place a mild full-support assumption on bidders' beliefs over their rivals' types. By using the full-support assumption, I show that if a bidder believes that her rivals play undominated strategies, then truthful reporting is her unique undominated best reply. The bidder does not overreport her budget because she believes that there is a positive probability of paying an amount that exceeds her actual budget if she overreports. As a robustness check, I show that we obtain similar results when we relax the assumption that bidders have hard budgets. In particular, I consider the case where bidders have continuous utility functions and get high (but finite) disutility from spending an amount that exceeds their budget.

The assumption of full-support beliefs is general enough to nest cases where bidder types

[2]Bhattacharya et al. also suggest that the auctioneer can use lotteries to elicit information on private budgets. However, such payment schemes are rarely seen in practice and lead to violations of bidders' ex-post individual rationality.

are $i.i.d.$, correlated, or do not satisfy a common prior assumption. The assumption is similar to the full-support assumption used by Ausubel (2004) in his paper defining the clinching auction. There are no restrictions on bidders' higher-order beliefs. Thus, the proxy clinching auction is detail-free, and robust in the sense of Wilson (1987).

The second difference from the prior literature is that I use a weaker solution concept. DLN only consider mechanisms that are dominant strategy implementable. I do not claim that bidders have a dominant strategy to report their types' truthfully in the proxy clinching auction. Instead, I show that underreporting values and/or overreporting budgets is weakly dominated, and truthful reporting is the prediction of two rounds of iterative elimination.

The rest of the paper proceeds as follows. Section 2 formalizes the auction setting, describes the proxy clinching auction, and lists some basic properties of the mechanism. Section 3 provides a motivating example. Section 4 characterizes bid behavior in the auction. Section 5 discusses efficiency. Section 6 considers two simple extensions. Section 7 concludes.

2 Model

2.1 Payoff Setting

The setting that I study is closely related to prior work studying efficient auction design with budgets.[3] My model description closely follows DLN. I later discuss how my results extend to the sale of an indivisible good.

A seller owns a divisible good that is in net supply one. There are N bidders, where bidder i has value v_i for each unit she wins, and budget b_i. I call $\theta_i = (v_i, b_i)$ bidder i's payoff type and assume that $\theta_i \in \Theta$, where $\Theta := \{\theta \in [0,1]^2 | 0 \leq b \leq v \leq 1\}$.[4] Thus, if $v_i = b_i$, then bidder i's willingness to pay for one unit of the good equals her ability to pay. If $v_i > b_i$, then bidder i faces a binding budget constraint. For ease of notation, I say $u_i(x, p)$ is the utility of bidder i with payoff type θ_i, when she wins $x \in [0,1]$ units and pays p, where

$$u_i(x, -p) = \begin{cases} v_i x - p & if\ p \leq b_i, \\ -\infty & if\ p > b_i. \end{cases}$$

Bidder i's payoff type θ_i is private information.

An outcome describes payments and the allocation of the good.

[3] See Borgs et al. (2005); DLN; and Hafalir, Ravi, and Sayedi (2012).

[4] The setting is isomorphic to one where $\Theta := \{[\underline{v}, \overline{v}]^2 | 0 \leq \underline{v} \leq b \leq v \leq \overline{v} < \infty\}$. We assume $[\underline{v}, \overline{v}] = [0,1]$ for simplicity. We could also allow for $b_i > v_i$; but such bidders have the same incentives as a bidder with $b_i = v_i$.

Definition 1. (Outcomes)

A (feasible) outcome $(x, p) \in [0,1]^N \times \mathbb{R}^N$ *is a vector of allocated quantities* $x_1, \ldots, x_N$ *and a vector of payments* $p_1, \ldots, p_N$ *with the property that* $\sum_{i=1}^{N} x_i \leq 1$.

The proxy clinching auction implements feasible outcomes that are (ex-post) individually rational, satisfy weak budget balance, and are Pareto efficient. An outcome is individually rational if all bidders receive non-negative payoffs.

Definition 2. (Individual rationality)

An outcome $(x, p) \in \mathbb{R}_+^{2N}$ *is individually rational if* $u_i(x_i, p_i) \geq 0 \ \forall i = 1, \ldots, N$.

Like DLN, I am interested in studying the implementation of outcomes that satisfy weak budget balance. DLN refer to this as a no positive transfers condition. Weak budget balance is an individual rationality constraint on the auctioneer, and it avoids trivializing the efficient implementation problem. If we do not impose a budget balance restriction, the auctioneer can pay all bidders a large amount and then hold a second price auction amongst the (now unconstrained) bidders. This is efficient and dominant strategy implementable, but violates budget balance.

Definition 3. (Weak budget balance)

An outcome $(x, p) \in \mathbb{R}_+^{2N}$ *satisfies the weak budget balance if* $\sum p_i \geq 0$.

Lastly, I study the implementation of (ex-post) Pareto efficient outcomes. My definition of Pareto efficiency is the same as the definition used by DLN.

Definition 4. (Pareto efficiency)

An outcome $\{(x_i, p_i)\}$ *is Pareto efficient if there does not exist a different outcome* $\{(x_i', p_i')\}$ *that makes all players better off,* $u_i(x_i', -p_i') \geq u_i(x_i, -p_i) \ \forall i$*, and gives weakly greater revenue* $\sum_i p_i' \geq \sum p_i$*, where at least one of the inequalities holds with a strict inequality.*

Bidder i has beliefs about the distribution of her rivals' values and budgets. I assume that bidder i's first-order beliefs satisfy a full-support assumption. Roughly speaking, this states that any realization of opponent payoff types is possible with some probability.

Assumption 1. (Full-support beliefs)

Bidder i *has a full-support prior if for any* $\theta_i \in \Theta$ *and any subset* $A \subset \Theta^{N-1}$ *where* A *has strictly positive Lebesgue measure* $\mu(A) > 0$*, then* $F_i(A|\theta_i) > 0$.

I do not require that bidders share a common prior, nor do I require that it is commonly known that all bidder's beliefs satisfy this condition. The full-support assumption only applies to bidders' first-order beliefs. We could go further, and explicitly model bidder i's higher-order beliefs, however, explicitly modeling higher-order beliefs can be intractable, and is unnecessary for the purpose of this paper.

DLN do not explicitly model bidders' beliefs. In their examples, bidders have an incentive to overreport their budgets. I show that when bidders have full-support beliefs, bidders no longer have the incentive to overreport budgets.

2.2 Description of the mechanism

The proxy clinching auction is a direct revelation mechanism where bidders report a payoff type (value and budget) to the auctioneer. Proxy bidders then play a clinching auction, like that described by Ausubel (2004). I adapt Ausubel's design to include for budgets by allowing bidders' demands to change depending upon the amount of money they spend. This is similar to the clinching auction studied in DLN.

Thus, for bidder i, a pure strategy a_i is a mapping from her type to her report, $a_i : \Theta \to \Theta$. Given the profile of reported bidder types $(\theta_1, \dots, \theta_N)$ the proxy clinching auction determines the number of units bidder i wins $\mathcal{Q}_i : \Theta^N \to [0, 1]$ and her payment $\mathcal{P}_i : \Theta^N \to [0, 1]$.

If all bidders report zero demand, the good is split equally among all bidders. That is, if the profile of reported types $(\theta_1, \dots, \theta_N)$ is such that $b_i = 0 \ \forall i$, then $\mathcal{Q}_i = \frac{1}{M}\mathbb{I}_{v_i>0}$ and $\mathcal{P}_i = 0 \ \forall i$, where M is the number of bidders who report strictly positive values. If only one bidder i reports positive demand, then i wins all units for a price of zero. That is, if $b_i > 0$, and $b_j = 0 \ \forall j \neq i$, then $\mathcal{Q}_i = 1$, $\mathcal{P}_i = 0$ and $\mathcal{Q}_j = \mathcal{P}_j = 0 \ \forall j \neq i$. In each case, we say the auction terminates at time 0.

The non-trivial cases occur when at least two bidders report non-zero demands. That is, there exists bidders i, j such that $i \neq j$ and $b_i, b_j > 0$. The proxy clinching auction starts at time 0. Time continuously increases until the auction terminates. The time t represents the marginal price of additional units at time t.

More formally, let $q_i(t) \in [0, 1]$ be the number of units clinched by bidder i at time t. Similarly, $p_i(t) \in \mathbb{R}_+$ is the total amount that bidder i has committed to pay at time t. I construct both q_i and p_i to be non-decreasing. At time $t = 0$, we set $q_i(0) = p_i(0) = 0 \ \forall i$. For convenience, denote $p_i^-(t) := \lim_{t' \to t^-} p_i(t')$ and $q_i^-(t) := \lim_{t' \to t^-} q_i(t')$.

Bidders continuously report their demands for additional units as time (i.e. the marginal price of additional units) increases. At time t, bidder i's demand for additional units is

$$d_i(t) = \begin{cases} \min\{1 - q_i^-(t), \frac{b_i - p_i^-(t)}{t}\} & if\ b_i \geq p_i^-(t)\ and\ v_i > t. \\ 0 & else \end{cases}$$

Bidder i demands the maximal number of additional units she can afford if the marginal price of additional units is below her value. She demands no additional units if the marginal price of units exceeds her value, or if she has spent in excess of her budget. Let z_i be the total number of units bidder i demands at time t, including the units she has already clinched up to time t, $z_i(t) = q_i^-(t) + d_i(t)$. I refer to z_i as the total demand of bidder i. Thus, the total demand of bidder i at time t includes the units bidder i has already clinched up to time t.

Each bidder i faces a residual demand curve s_i that is a function of other (proxy) bidders' reported demands and the quantity they have clinched,

$$s_i(t) = \begin{cases} 1 - \sum_{j \neq i} z_j(t) & if\ 1 \geq \sum_{j \neq i} z_j(t), \\ 0 & else. \end{cases}$$

If at time t, $d_i(t') > 0\ \forall t' < t$, then

$$q_i(t) = \min\{\sup_{t \leq t'} s_i(t), d_i(t) + \sup_{t < t'} s_i(t)\}.$$

That is, the supply curve determines the quantity that bidder i clinches, but we add the additional restrictions that (1) the quantity bidder i clinches is non-decreasing, and that (2) at time t, bidder i never clinches any more additional units than she demands. Thus, the amount bidder i has clinched at time t can never exceed her total demand for units at time t, $z_i(t)$.

If we reach some time t where $d_i(t) = 0$, then bidder i does not clinch any additional units. In particular, if we define $t^* = \sup\{t | d_i(t') > 0\ \forall t' \leq t\}$ as the first time bidder i has zero demand for additional units, then bidder i clinches no additional units following time t^*.

$$q_i(t) = q_i(t^*)\ if\ t > t^*.$$

Therefore, having zero demand is equivalent to dropping out of the auction.

Bidder i pays t per unit for any additional units that she clinches at time t. Thus,

$$p_i(t) = q_i(t)t - \int_0^t q_i(s)ds.$$

The auction terminates at time τ, where τ is the first time the quantity of unclinched units up to τ (weakly) exceeds the demands for additional units, $1 - \sum_{i=1}^N q_i^-(t) \geq \sum_{i=1}^N d_i(t)$. Or equivalently, time τ is the first time when all bidders' total demands fall below the supply of units, $1 \geq \sum_{i=1}^N z_i(t)$. The terminating time τ is a function of the full profile of bidder reports; I suppress notation for succinctness.

If $1 = \sum_{i=1}^N z_i(\tau)$ at time τ, then each bidder wins $\mathcal{Q}_i = z_i(\tau)$ units and pays $\mathcal{P}_i =$

$p_i^-(\tau) + \tau d_i(\tau)$.

If $1 > \sum_{i=1}^N z_i(\tau)$, we use a rationing rule. Note that $\lim_{t' \to \tau^-} \sum_{i=1}^N z_i(t') \geq 1$. If not, the auction would terminate at a time earlier than τ. Let $H := \lim_{t' \to \tau^-} \sum_{i=1}^N z_i(t')$ and $L := 1 > \sum_{i=1}^N z_i(\tau)$. By construction, $H \geq 1 \geq L$. Thus, in this case where $H > L$ and we use the rationing rule, bidder i wins $\mathcal{Q}_i$ units, where

$$\mathcal{Q}_i := \frac{1-L}{H-L}\left(\lim_{t \to \tau^-} z_i(t)\right) + \frac{H-1}{H-L} z_i(\tau).$$

She pays $\mathcal{P}_i$, where

$$\mathcal{P}_i = p_i^-(\tau) + \tau\left(\mathcal{Q}_i - q_i^-(\tau)\right).$$

If z_i is left continuous at τ, the above expression simplifies to say that bidder i wins $z_i(\tau)$ units and pays $p_i^-(\tau) + \tau d_i(\tau)$. When z_i has a left discontinuity at τ, bidder i wins between $\lim_{t \to \tau^-} z_i(t)$ and $z_i(\tau)$ units. The precise number of units is a weighted sum of the two quantities. The weights are chosen to ensure feasibility. Bidder i pays τ for any additional units won at time τ. By construction, $\sum_{i=1}^N \mathcal{Q}_i = 1$.

2.3 Basic properties of the proxy clinching auction

Lemma 1 summarizes six properties that follow from the construction of the proxy-clinching auction.

Lemma 1. (Properties of the proxy clinching auction)
Consider any profile of bidder reports $(\theta_1, \ldots, \theta_N)$ in the proxy clinching auction. Suppose that at least two bidders report positive demands $b_i, b_j > 0$, for some $i, j \in \{1, \ldots, N\}$. Then,

(1) $\tau > 0$.

(2) $z_i(t)$ is non-increasing in t over $(0, \tau)$.

(3) s_i is non-decreasing in t, and $q_i(t) = s_i(t)$ $\forall t \in (0, \tau)$.

(4) $\tau \leq 1$.

(5) If $d_i(t) = 0$ for some $t < \tau$, then $\mathcal{Q}_i = \mathcal{P}_i = 0$.

(6) If $t \in (0, \tau)$, then $p_i(t) < b_i$; and $\mathcal{P}_i \in [0, b_i]$ $\forall i$.

The first point states that if at least two people state positive demands, then the proxy clinching auction will not terminate at time 0. The second point states that a bidder's total demand for units weakly decreases as time increases. A direct implication of this point is that the number of units bidder i has clinched at time t is the number of units that are neither clinched nor demanded by other bidders at time t. The fourth point states that the auction terminates before the marginal price of additional units strictly exceeds all bidders' values. The fifth point states that a bidder who reports zero demand before the auction terminates drops out of the auction. That is, she wins no units and pays nothing. The sixth point states that the proxy clinching auction never requires bidders to make a payment that exceeds their stated budget.

3 A Motivating Example

The purpose of this example is to illustrate the intuition for DLN's impossibility result and to motivate my main result: that the proxy clinching auction implements a Pareto efficient allocation when bidders have full-support beliefs.

Suppose that there are two bidders who compete in the proxy clinching auction. Bidder 1 has type $\theta_1 = (\frac{3}{4}, \frac{2}{3})$ and bidder 2 has type $\theta_2 = (1, \frac{1}{2})$. Thus, both bidders are budget constrained. If both bidders report their types truthfully, then bidder 1 wins $\mathcal{Q}_1 = .331$ units and she pays $\mathcal{P}_1 = .201$. Bidder 2 wins $\mathcal{Q}_2 = .669$ units and pays $\mathcal{P}_2 = \frac{1}{2}$.

However, conditional upon knowing her rival's reported type, bidder 1 has a profitable deviation. To see this, suppose that bidder 1 reports her type to be $\tilde{\theta}_1 = (\frac{3}{4}, \frac{3}{4})$ and bidder 2 reports her type truthfully. Then, $\mathcal{Q}_1 = \frac{1}{3}$, $\mathcal{Q}_2 = \frac{2}{3}$, $\mathcal{P}_1 = .202$, and $\mathcal{P}_2 = .5$. By overreporting her budget, bidder 1 increases her payoff because

$$u_1(\frac{1}{3}, 0.202) > u_1(0.331, 0.201).$$

The incentive to overreport budgets follows from the construction of the clinching auction and the presence of budgets. If bidder 1 reports her true (lower) budget, bidder 2 is able to clinch units at an earlier time, when the price of units is low. This means, bidder 2's total demand for units is higher, because bidder 2 is able to clinch units at a lower price and use less of her budget to acquire the same number of units. Because bidder 2 has a higher total demand, this means that the residual demand left to bidder 1 is smaller. The residual demand is the quantity that bidder 1 clinches. Thus, bidder 1 must wait longer to clinch the same number of units. That is, bidder 1 pays a higher marginal price to acquire the same number of units.

This is the motivation behind DLN's impossibility theorem. They show that when bidders' budgets are public, the clinching auction is the unique efficient mechanism, but with private budgets, bidders have an incentive to overreport their budget. Thus, there is no mechanism that is dominant strategy incentive compatible and Pareto efficient when budgets are private.

In the above example, bidder 1 has an incentive to overreport because she knows that her payment will not exceed her budget. Suppose instead, that bidder 2 reported that her type was $(\frac{8}{11}, \frac{8}{11})$. If bidder 1 still reports that her type is $(\frac{3}{4}, \frac{3}{4})$, now neither bidder is (reportedly) budget constrained. In this case, the outcome of the proxy clinching auction is equivalent to the outcome of the second price auction. This means bidder 1 wins all units at a price of $\frac{8}{11}$. However, bidder 1's payment exceeds her (actual) budget of $\frac{2}{3}$. Therefore, bidder 1 would have been better off if she had instead truthfully reported her type.

Thus, a bidder may have incentives to overreport her budget if she knows her rivals' bids. However, when her rivals' bids are sufficiently dispersed, she does better by truthfully reporting her budget. Thus, if the bidder believes that this is a positive probability that she will pay in excess of her budget by overreporting her budget, she will have an endogenous incentive to not overreport. I show that this is the case when a bidder has full-support beliefs, and her rivals play undominated strategies. In order to show this, I first obtain bounds on bid behavior by eliminating dominated strategies. As a robustness check, I show that we get similar results even if a bidder does not have a hard budget.

4 Bid Behavior in the proxy-clinching auction

While DLN show that truthful reporting is not a dominant strategy in the proxy clinching auction, we are still able to bound bid behavior by eliminating dominated strategies.

Proposition 1 shows that if bidder i has type $\theta_i = (v_i, b_i)$, then reporting $(v, b) \in \Theta$ where $v > v_i$ is weakly dominated by reporting $(v_i, \min\{v_i, b\})$. That is, holding her reported budget fixed, she earns a weakly lower payoff by overreporting her value. The intuition mirrors the intuition for why bidders do not overreport values in the second price auction. If bidder i reports value $v > v_i$, then any additional units won after time v_i decreases her payoff. Bidder i does better by reporting value v_i and only winning the units she had clinched prior to time v_i, when the marginal price of units is less than her value.

Proposition 1. (Overreporting values is weakly dominated)
Overreporting values is a weakly dominated strategy. Specifically, if bidder i has type (v_i, b_i) where $b_i > 0$, then reporting type $(v, b) \in \Theta$, where $v > v_i$, is weakly dominated by reporting $(v_i, \min\{v_i, b\})$.

Proposition 1 gives an upper bound on bidders' reported values. Similarly, Proposition 2 gives a lower bound on bidders' reported budgets. Specifically, Proposition 2 shows that simultaneously underreporting values and budgets is weakly dominated by truthful reporting.

Proposition 2. (Underreporting both values and budgets is weakly dominated)
If bidder i has type (v_i, b_i), then reporting type $(v, b) \in \Theta$, where $v \leq v_i$ and $b < \min\{v, b_i\}$, is weakly dominated by reporting (v_i, b_i).

The proof of Proposition 2 is broken into three Lemmas (2-4). Lemmas 2 and 3 compare bidder i's payoff when she reports $(v, b) \in \Theta$ with her payoff when she reports $(v, \min\{b_i, v\})$, where $b < \min\{b_i, v\}$. That is, fixing bidder i's reported value, I compare her payoff from reporting a higher budget with her payoff from reporting a lower budget.

The superscript ℓ denotes when bidder i reports value v and the lower budget b. The superscript h denotes when bidder i reports value v and a higher budget equal to $\min\{v, b_i\}$, with $b < \min\{v, b_i\}$. Lemma 2 shows that given the reports of bidder $j \neq i$, bidder i clinches more units at time t by reporting a higher budget than she does by reporting a lower budget. Or equivalently, $q_i^h(t) \geq q_i^\ell(t)$.

Lemma 2. (Bidders clinch more objects by reporting a higher budget)
If $t \in (0, \min\{\tau^\ell, \tau^h)$, then $q_i^h(t) \geq q_i^\ell(t)$.

The intuition for the Lemma is straightforward. At any time t, if bidder i reports the higher budget, she clinches more units than she would if she were to report the lower budget. By reporting the higher budget, bidder i has weakly greater demand for units. When she reports the weakly greater demand, she clinches a weakly greater number of units at any time t.

I use Lemma 2 to show that reporting $(v, \min\{b_i, v\})$ weakly dominates reporting (v, b) for any $b < \min\{v, b_i\}$, when $v \leq v_i$. In other words, simultaneously underreporting values and budgets is weakly dominated by truthfully reporting your budget while keeping your reported value unchanged.

To prove this, I use Lemma 2 to show that when bidder i reports a higher budget, she wins a weakly greater number units, $\mathcal{Q}_i^h \geq \mathcal{Q}_i^\ell$. In addition, I show that bidder i pays a lower price to win her first $\mathcal{Q}_i^\ell$ units by reporting the higher budget. This is because $q_i^h(t) \geq q_i^\ell(t)\ \forall t \in (0, \min\{\tau^\ell, \tau^h)$. That is, bidder i wins her first $\mathcal{Q}_i^\ell$ units at an earlier time when she reports the high budget versus the low budget. Thus, her marginal price of each unit is lower when she reports the higher budget. These results combine to show that reporting $(v, \min\{b_i, v\})$ weakly dominates reporting (v, b) for any $b < \min\{v, b_i\}$, when $v \leq v_i$. This argument is proven formally in Lemma 3.

Lemma 3.
If bidder i has type (v_i, b_i), then reporting $(v, b) \in \Theta$, where $v \leq v_i$ and $b < \min\{v, b_i\}$, is weakly dominated by bidding $(v, \min\{v, b_i\})$.

I finish the proof of Proposition 2, by showing that reporting $(v, \min\{v, b_i\})$ is weakly dominated by reporting (v_i, b_i) if $v < v_i$. That is, bidder i gets a weakly greater payoff from truthful reporting than she does from any report where she underreports her value and truthfully reports her budget (subject to the constraint that her reported budget does not exceed her reported value). This is stated formally in Lemma 4.

The intuition behind the proof mirrors the standard argument used to show that underreporting values is weakly dominated in a second price auction. If she reports a value v less than her true value v_i, then at any time $t \in (v, v_i)$ she demands no additional units. However, bidder i increases her payoff by clinching additional units at a marginal price $v < t < v_i$. In addition, bidder i never exceeds her actual budget by reporting the higher value, as $\mathcal{P}_i \leq b_i$ under truthful reporting.

Lemma 4.
If bidder i has type (v_i, b_i), then reporting $(v, \min\{v, b_i\})$, where $v < v_i$, is weakly dominated by bidding (v_i, b_i).

Lemmas 3 and 4 combine to prove Proposition 2. For any profile of bids and valuations reported by bidders $j \neq i$, θ_{-j}, Lemma 3 states that reporting $(v, b) \in \Theta$ where $v \leq v_i$ and $b < \min\{v, b_i\}$ gives a weakly lower payoff than reporting $(v, \min\{v, b_i\})$. In addition, Lemma 4 shows that reporting $(v, b_i) \in \Theta$, where $v < v_i$, gives a weakly lower payoff than reporting truthfully (v_i, b_i). Thus, truthful reporting gives a weakly greater payoff than reporting $(v, b) \neq (v_i, b_i)$ where $v \leq v_i$ and $b \leq b_i$.

I describe $\mathcal{U}(\theta_i) \subset \Theta$ as the set of all undominated strategies for bidder i. Thus far, our results have shown that $a \in \mathcal{U}(\theta_i)$ only if

$$a = (v, b) \text{ where } (v, b) = (v_i, b_i) \text{ or } v < v_i \text{ and } b_i < b \leq v \leq v_i. \tag{1}$$

That is, overreporting values and underreporting budgets are weakly dominated. In addition, if bidder i reports her budget truthfully, underreporting her value is also weakly dominated. This is displayed in Figure 1.

When bidder i has full-support first-order beliefs, the bounds on bid behavior from equation (1) imply that truthful reporting is the unique best reply to any undominated strategy profile of bidders $j \neq i$. If bidder i overreports her budget, full-support first-order beliefs imply that there is a positive probability that her payment exceeds her budget. Specifically, full-support first-order beliefs imply that if bidder i reports her type to be (v, b) where $b > b_i$

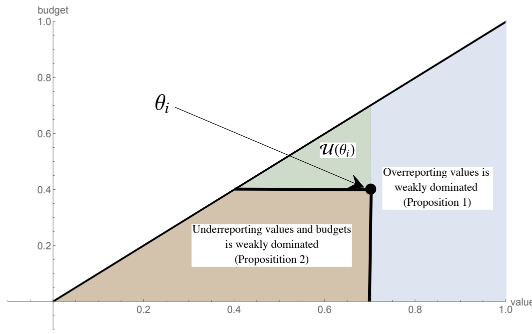

Figure 1: Undominated strategies

and $b_i < v \leq v_i$, then there is a positive probability that she pays $\mathcal{P}_i \in (b_i, v)$ if her opponents play undominated strategies. This is proved in Proposition 3.

Proposition 3. (Truthful reporting is the unique undominated best reply)
Truthful reporting is the unique undominated best reply to any undominated strategy profile of bidders $j \neq i$.

Proposition 3 shows that two rounds of iterative elimination of weakly dominated strategies predict that bidders report their types truthfully. Thus, if bidders report their types truthfully, we can say that if the auction ends at time τ, all bidders receive their total demand for units at time τ.

It is commonly known that iterative elimination of weakly dominated strategies can yield multiple predictions. However, multiplicity in not an issue in this auction setting. Truthful reporting is never eliminated under any iterative procedure as truthful reporting is a strict best reply to truthful reporting. Insincere bidding is a best reply if and only if a bidder believes her rivals play dominated strategies.

Corollary 1. (Properties of the allocation)
Given any profile of bidder types $(\theta_1, \ldots, \theta_N)$, if bidder i reports her type truthfully, and the auction terminates at time τ, then

$$\mathcal{Q}_i = \begin{cases} 0 & if \;\; \tau > v_i \\ [0, \lim_{t \to \tau^-} z_i(\tau)] & if \;\; \tau = v_i \\ z_i(\tau) & if \;\; v_i > \tau \end{cases}.$$

This follows directly from our prior results. If $\tau > v_i$, then $\exists t \in (0, \tau)$ such that $d_i(t) = 0$.

Lemma 1 then implies that $\mathcal{Q}_i = \mathcal{P}_i = 0$. If $\tau < v_i$, then

$$z_i(t) = q_i^-(t) + d_i(t) = \min\{1, \frac{b_i + \int_0^\tau q_i(s)ds}{\tau}\} \; \forall t \in [0, \tau].$$

Since $z_i(\tau) = \lim_{t\to\tau^-} z_i(t)$ for all i, then $\mathcal{Q}_i = z_i(\tau)$ by construction. If $\tau = v_i$, then $z_i(\tau) \neq \lim_{t\to\tau^-} z_i(t)$ because $d_i(t)$ is discontinuous at τ. Thus, bidder i wins some amount between 0 and $\lim_{t\to\tau^-} z_i(t)$.

5 Efficiency

Proposition 3 shows that bidders truthfully report their types in the proxy clinching auction. I use the first welfare theorem to show this gives a Pareto efficient outcome.

Suppose that the outcome of the proxy clinching auction is such that bidder i wins $\mathcal{Q}_i$ units and pays $\mathcal{P}_i$, and the auction terminates at time τ. Note that $\mathcal{P}_i = \mathcal{Q}_i\tau - \int_0^\tau q_i(s)ds$.

I invoke the first welfare theorem because the outcome of the proxy clinching auction is a Walrasian equilibrium of a two commodity endowment economy with $N+1$ agents. The two commodities being traded are money and the divisible good. Each of the $N+1$ agents has endowments that are identical to the outcome of the proxy-clinching auction. Specifically, bidder i is endowed with $\mathcal{Q}_i$ units of the good and $b_i - \mathcal{P}_i$ units of money. I assume i has preferences

$$U_i(q, m) = \begin{cases} qv_i + m & if\ q \in [0, 1] \\ v_i + m & if\ q > 1 \end{cases}.$$

Note that $U_i(q, b_i - \mathcal{P}_i) = u_i(q, -\mathcal{P}_i)$. The auctioneer is agent 0. She is endowed with no units of the good and $\sum \mathcal{P}_i$ units of money. Her preferences are $U_0(q, m) = m$. I show that the endowment economy has a Walrasian equilibrium with no trade, where the price of money is 1 and the price of the good is τ. Thus, the initial endowment of goods is Pareto efficient, as is the outcome of the proxy clinching auction.

Proposition 4. (The proxy clinching auction is efficient)
The proxy clinching auction implements a Pareto efficient outcome.

The intuition behind the proof is straightforward. The auction terminates at time τ. Prior to time τ, bidder i may have already clinched some units at a price lower than τ. This discount is equal to $\int_0^\tau q_i(s)ds$. Thus, it is as though bidder i has $b_i + \int_0^\tau q_i(s)ds$ to spend. Bidder i's Marshallian demand M_i for units of the good when she has wealth $b_i + \int_0^\tau q_i(s)ds$

and the price of the good is τ is

$$M_i = \begin{cases} 0 & if\ \tau > v_i \\ [0, \min\{1, \frac{b_i + \int_0^\tau q_i(s)}{\tau}\}] & if\ \tau = v_i \\ \min\{1, \frac{b_i + \int_0^\tau q_i(s)}{\tau}\} & if\ \tau < v_i \end{cases}.$$

This is the quantity that bidder i wins in the proxy clinching auction. Thus, the initial endowments are a Walrasian equilibrium, and the outcome of the proxy clinching auction is a Walrasian equilibrium.

6 Extensions

6.1 Continuous Utility Function

The results in the prior sections leverage the assumption that bidders get infinite disutility from exceeding their budget. Bidders do not overreport their budget because any overreport gives a positive probability of obtaining infinite disutility. As a robustness check, I show that we obtain similar results when bidders have continuous utility functions, and receive high (but finite) marginal disutility from spending in excess of their budget.

Specifically, assume that bidder i, with type $\theta_i = (v_i, b_i)$, has preferences

$$u(x, -p, \theta_i) = \begin{cases} xv_i - p & if\ p \leq b_i \\ xv_i - b_i - \varphi(p - b_i) & if\ p > b_i \end{cases},$$

where $\varphi > 1$. A bidder's marginal disutility of spending a dollar over her budget is $\varphi > 1$.

The implications of Propositions 1 and 2 hold when we assume bidders have continuous utility functions (φ is finite). That is, overreporting values and underreporting budgets are weakly dominated.

To show that the implications of Proposition 2 hold, the proof is similar to the proof that bidders do not overreport values in the second price auction. Holding a bidder's reported budget fixed, if a bidder overreports her value, there is a possibility that she wins additional units at a price per unit that exceeds her value. This is avoided if she reports her value truthfully.

The implications of Proposition 2 hold, because the proof of Proposition 2 is unchanged when we assume φ is finite. This is because a bidder's payment does not exceed her budget if she reports her type truthfully or if she simultaneously underreports her value and budget. Therefore, the set of undominated strategies is unchanged when φ is finite.

Corollary 2. *The report* $(v, b) \in \mathcal{U}(\theta_i)$ *only if*

$$(v, b) \text{ is such that } (v, b) = (v_i, b_i) \text{ or } b_i < b \leq v \leq v_i.$$

If bidder i reports the undominated strategy $(v, b) \neq (v_i, b_i)$, then it is necessarily the case that she overreports her budget. However, if (1) bidder i believes that her rivals' play undominated strategies, and (2) bidder i receives sufficiently large disutility from spending in excess of her budget, then truthful reporting gives greater expected utility than reporting (v, b). If bidder i overreports her budget, then full-support beliefs imply there is a positive probability that she pays an amount that exceeds her actual budget. When φ is sufficiently large, the disutility associated with exceeding the budget is greater than any possible gain that could occur from misreporting. Thus, the expected payoff from reporting (v, b) is negative. At the same time, truthful reporting guarantees that bidder i receives a non-negative payoff. Thus, the report (v, b) is eliminated in the second round of iterative elimination of weakly dominated strategies, when φ is sufficiently large.

Proposition 5. *Consider any report* $(v, b) \neq (v_i, b_i)$*, where* $(v, b) \in \mathcal{U}(\theta_i)$*. Suppose all bidders play undominated strategies and bidder* i *has full-support first-order beliefs. If* α *is sufficiently large, the expected utility of truthful reporting is strictly greater than the expected utility of reporting* (v, b)*.*

6.2 Indivisible goods

My results can also be used to study a setting where the auctioneer sells a single indivisible good. The auctioneer uses the proxy clinching auction to sell probabilities of winning the indivisible good as though they were shares of a divisible good in net supply one. Thus, the proxy clinching auction determines each bidder's probability of winning the good and her payment. Bidders' payments have an all-pay structure. That is, a bidder pays the same amount whether or not she wins the good. Bidder payments are increasing in the bidder's probability of winning.

More formally, consider the proxy clinching auction for a divisible good where bidders report types (θ_i, θ_{-i}). Suppose that bidder i wins quantity $\mathcal{Q}_i$ and pays $\mathcal{P}_i$. Then, in the indivisible good setting, if bidders report types (θ_i, θ_{-i}) in the proxy clinching auction, bidder i wins the good with probability $\mathcal{Q}_i$, and pays $\mathcal{P}_i$, independent of whether she wins the good. Thus, bidder i has expected utility u_i, where

$$u_i(\mathcal{Q}_i, \mathcal{P}_i) = \begin{cases} v_i \mathcal{Q}_i - \mathcal{P}_i & if \ \mathcal{P}_i \leq b_i, \\ -\infty & if \ \mathcal{P}_i > b_i. \end{cases}$$

This is the same as the utility of a bidder with type (v_i, b_i) who wins $\mathcal{Q}_i$ units and pays $\mathcal{P}_i$ in the divisible good setting. Thus, a bidder's incentive to report her type is the same as it is in the proxy clinching auction for divisible goods.

Corollary 3. (Truthful reporting is the unique undominated best response)
If a bidder has full-support first-order beliefs, truthful reporting is the only strategy that survives two rounds of iterative deletion of weakly dominated strategies.

Thus, the proxy clinching auction can be solved using two rounds of iterative elimination. I modify my earlier definition of an outcome to allow for indivisibility. Let $\mathbb{A}$ be the set of all feasible assignments, where

$$\mathbb{A} := \{a | a \in \{0,1\}^N \text{ and } \sum_{i=1}^{N} a_i \leq 1\},$$

where $a_i = 1$ if bidder i is given the object. A (deterministic) outcome ϕ specifies both transfers and a feasible assignment: $\phi \in \mathbb{A} \times \mathbb{R}^N$. I define $\Phi := \mathbb{A} \times \mathbb{R}^N$ as the set of all (deterministic) outcomes. Thus, a (probabilistic) outcome α is an element of $\Delta(\Phi)$.

An outcome α is (ex-post) Pareto efficient if, conditional on bidder types $(\theta_1, \ldots, \theta_N)$, there is no other outcome that gives weakly greater expected revenues and makes all bidders weakly better off in expectation. That is, conditional upon knowing all bidders' private information, we cannot increase one bidder's expected utility without necessarily decreasing revenue or lowering another bidder's expected utility. Pareto efficiency is defined formally in Definition 5. This is the definition of ex-post Pareto efficiency used by Holmstrom and Myerson (1983). The notation $\mathbb{E}_\alpha[v_i, b_i]$ denotes the expected utility of a bidder i with valuation v_i and budget b_i in outcome $\alpha \in \Delta(\Phi)$. Similarly, $\mathbb{E}_\alpha[P]$ denotes the expected total payment collected by the auctioneer.

Definition 5. (Pareto efficient)
An outcome $\alpha \in \Delta(\Phi)$, is Pareto efficient if $\nexists \alpha' \in \Delta(\Phi)$ such that

$$\mathbb{E}_{\alpha'}[v_i, b_i] \geq \mathbb{E}_\alpha[v_i, b_i] \ \forall i = 1, \ldots, N,$$

and

$$\mathbb{E}_{\alpha'}[P] \geq \mathbb{E}_\alpha[P],$$

where at least one of the above inequalities holds strictly.

Corollary 4. (The proxy clinching auction is efficient)
The outcome of the proxy clinching auction is Pareto efficient.

The proof of this Corollary follows from our prior results. Let $\mathcal{A} \subset \Delta(\Phi)$ be the set of all outcomes where bidders pay a constant amount to the auctioneer, i.e. the set of all outcomes with all-pay payment schemes. By construction the outcome of the proxy clinching auction is an element of $\mathcal{A}$.

Consider an outcome $\alpha \in \Delta(\Phi)$ where bidder i wins with probability q_i and pays p_i in expectation, and the auctioneer is paid $\sum_{i=1}^{N} p_i$ in expectation. Then, there is a corresponding outcome $\tilde{\alpha} \in \mathcal{A}$, where bidder i wins with probability q_i and pays p_i with certainty. The auctioneer collects equal expected payments under α and $\tilde{\alpha}$. Thus, outcome $\tilde{\alpha}$ makes all bidders weakly better off and gives equal revenue. This holds because

$$\mathbb{E}_{\tilde{\alpha}}[v_i, b_i] = q_i v_i - p_i \geq \mathbb{E}_{\alpha}[v_i, b_i] \ if \ b_i \geq p_i,$$

and

$$\mathbb{E}_{\tilde{\alpha}}[v_i, b_i] = \mathbb{E}_{\alpha}[v_i, b_i] = -\infty \ if \ b_i < p_i.$$

Thus, if outcome $\tilde{\alpha}$ is Pareto dominated by some outcome $\beta \in \Delta(\Phi)$, then there exists a corresponding outcome $\tilde{\beta} \in \mathcal{A}$ that also Pareto dominates outcome α.

A direct implication of Proposition 4 is that for any profile of bidder types $(\theta_1, \ldots, \theta_N) \in \Theta^N$, the outcome of the proxy clinching auction η is never Pareto dominated by some outcome $\tilde{\eta} \in \mathcal{A}$. Thus, $\nexists \tilde{\eta} \in \mathcal{A}$ such that

$$\mathbb{E}_{\tilde{\eta}}[v_i, b_i] \geq \mathbb{E}_{\eta}[v_i, b_i], \ \forall i$$

and

$$\mathbb{E}_{\tilde{\eta}}[P] \geq \mathbb{E}_{\eta}[P],$$

where at least one of the above holds strictly.

Because there does not exists a $\tilde{\eta} \in \mathcal{A}$ that Pareto dominates η, there is no outcome $\gamma \in \Delta(\Phi)$ that Pareto dominates η. This holds because if there was an outcome γ that did Pareto dominate η, there there would exist a corresponding $\tilde{\gamma} \in \mathcal{A}$ that also Pareto dominates η. However, since there is no $\tilde{\gamma} \in \mathcal{A}$ that exists, it follows that the outcome of the proxy clinching auction η is Pareto efficient.

7 Conclusion

This paper studies the design of efficient auctions when bidders have private values and private budgets. When bidders have budgets, the VCG mechanism loses its desired incentive and efficiency properties. I show that a clinching auction played by proxy bidders can be considered a useful second-best solution to the efficient auction design problem. The auction is not dominant strategy implementable, but it can be solved using two rounds of iterative elimination. This contrasts with prior work that shows that there is no dominant strategy incentive compatible mechanism that yields an efficient outcome when bidders have private values and private budgets. I obtain my contrasting results, by (1) using the weaker solution concept of iterative elimination, and (2) imposing a full-support assumption on bidders' first-order beliefs. The auction is efficient, because the outcome can be supported as the Walrasian equilibrium outcome of an endowment economy.

There are a number of related ideas that follow from this work. First, the model studied here could be extended to a case where bidders do not have constant marginal values for additional units. The intuition for not overreporting budgets should still hold with a full-support belief on opponents' types. Second, one could move from a private value setting, to an interdependent value setting. This paper extends Ausubel's (2004) clinching auction to the setting where bidders have budgets. With interdependent values, we could similarly adapt Perry and Reny's (2005) ascending auction to include budget constraints. In addition, it would be interesting the compare the performance of the clinching auction with other mechanisms under alternative welfare criteria. For example, recent work by Che, Gale, and Kim (2013) and Richter (2013) evaluate mechanisms in terms of utilitarian surplus, but both consider settings with a continuum of bidders. Finally, it may be useful to study how full-support beliefs and iterative elimination can be used to characterize behavior in other mechanism design environments where budgets are a relevant feature.

Appendix

Proof of Lemma 1, Part 1. By assumption, there exists at least two bidders i, j with $b_i, b_j > 0$. Let $\delta = \min\{b_i, b_j\}$.

At any time $t < \delta$, I show that $z_i(t) = 1$. Note that, $p_i^-(t) \leq tq_i^-(t) \leq t < b_i$ where the second inequality holds because $q_i(t) \in [0,1]$. If $q_i^-(t) = 1$, this holds trivially, and if $q_i^-(t) < 1$, then $p_i(t) \leq tq_i(t) \leq t < b_i$ and $d_i(t) = 1 - q_i^-(t)$, because $\frac{b_i - p_i^-(t)}{t} \geq \frac{b_i - tq_i^-(t)}{t} > 1 - q_i^-(t) = d_i(t)$. Thus, $z_i(t) = q_i^-(t) + d_i(t) = 1$. Similarly, $z_j(t) = 1 \; \forall t < \delta$.

Thus, $\sum_{i=1}^N z_i(t) \geq 2 > 1$ for all $t < \delta$. This implies the auction does not terminate until at or after time δ, and $\tau \geq \delta > 0$.

Part 2. I consider three cases. First, suppose that at time t, $d_i(t) = 0$. If $d_i(t) = 0$, then $p_i^-(t) \geq b_i$ and/or $t \geq v_i$. For all $t' > t$, $q_i^-(t') = q_i^-(t)$ by construction. Similarly, $p_i^-(t') = p_i(t) \; \forall t' > t$. Thus, $d_i(t') = 0 \; \forall t' > t$, and $z_i(t) = q_i^-(t) = q_i^-(t') = z_i(t')$.

Second, suppose that at time t, $d_i(t) = 1 - q_i^-(t) > 0$. Then $z_i(t) = 1 \geq z_i(t') \; \forall t \in (t, \tau)$, since $z_i(t') \leq 1 \; \forall t' \in (0, \tau)$ by construction.

Finally, suppose that at time t, $d_i(t) \in (0, 1 - q_i^-(t))$. Thus, $d_i(t) = \frac{b_i - p_i^-(t)}{t}$.

First, I show that at any time $t' > t$, $p_i^-(t') \leq b_i$. I use proof by contradiction. If $\exists t'$ such that $p_i^-(t') > b_i$, then, $\exists \tilde{t}$ is such that $t < \tilde{t} \implies p_i^-(t) \leq b_i$, and $t > \tilde{t} \implies p_i^-(t) > b_i$, because p_i is non-decreasing. At time $\tilde{t}$, $d_i(\tilde{t}) \leq \frac{b_i - p_i^-(\tilde{t})}{\tilde{t}}$. In addition, $q_i(\tilde{t}) \leq q_i^-(\tilde{t}) + d_i(\tilde{t}) \implies p_i(\tilde{t}) \leq p_i^-(\tilde{t}) + \tilde{t} d_i(\tilde{t}) \leq b_i$. Recall, we define t^* as the last time where bidder i demands additional units $t^* = \sup\{t | d_i(t') \geq 0 \; \forall t' \leq t\}$. Since $d_i(t) = 0 \; \forall t > \tilde{t}$, then $t^* \leq \tilde{t}$. Thus $q_i(t^*) = q_i(t') \; \forall t' > t^*$ by construction. It follows that $p_i(t^*) = p_i(t') \; \forall t > t^*$, and $p_i(t^*) \leq b_i$. This contradicts the assumption that there exists a time t' where $p_i(t') > b_i$.

Thus, if $t' > t$, then $p_i^-(t') \geq t(q_i^-(t') - q_i^-(t)) + p_i^-(t)$. This implies,

$$d_i(t') \leq \frac{b_i - p_i^-(t')}{t'} \leq \frac{b_i - p_i^-(t')}{t'} \leq \frac{b_i - \big(t(q_i^-(t') - q_i^-(t)) + p_i^-(t)\big)}{t'}$$

and,

$$d_i(t') \leq \frac{b_i - \big(t(q_i^-(t') - q_i^-(t)) + p_i^-(t)\big)}{t'} \leq \frac{b_i - p_i(t)}{t} + q_i^-(t) - q_i^-(t') = z_i(t) - q_i^-(t').$$

Thus,

$$d_i(t') \leq z_i(t) - q_i^-(t') \implies d_i(t') + q_i^-(t') = z_i(t') \leq z_i(t).$$

Part 3. First, I show that $q_i(t) \leq s_i(t)$. Recall that $s_i(t) = \max\{1 - \sum_{j \neq i} z_j(t), 0\}$. Since $z_j(t)$ is non-increasing in t, for all j, then $s_i(t)$ is non-decreasing. Thus, if $d_i(t') > 0 \; \forall t' < t$, then $q_i(t) = \min\{s_i(t), \lim_{t' \to t^-} s_i(t') + d_i(t)\}$ and $q_i(t) \leq s_i(t)$. If $d_i(t') = 0$ for some $t' < t$, then $q_i(t) = q_i(t^*)$, where t^* is defined as $t^* = \sup\{t | d_i(t') > 0 \ \forall t' < t\}$. Thus, $q_i(t) = q_i(t^*) \leq s_i(t^*) \leq s_i(t)$ since $s_i(t)$ is non-decreasing. Thus, $q_i(t) \leq s_i(t)$.

The remainder of the proof proceeds by contradiction. Suppose there exists a time $t \in (0, \tau)$ such that $q_i(t) < s_i(t)$.

I consider two cases. First, suppose t is such that $d_i(t') > 0 \ \forall t' < t$, then $q_i(t) = \min\{s_i(t), \lim_{t' \to^- t} s_i(t') + d_i(t)\} \implies q_i^-(t) = \lim_{t' \to^- t} s_i(t')$. Note that $q_i(t) = \lim_{t' \to^- t} s_i(t') + d_i(t) < s_i(t)$ because $q_i(t) = \min\{s_i(t), \lim_{t' \to^- t} s_i(t') + d_i(t)\} < s_i(t)$. Thus, $q_i(t) = z_i(t) < s_i(t)$. Since $s_i(t) > q_i(t) \geq 0$, then $s_i(t) = 1 - \sum_{j \neq i} z_j(t)$ and

$$z_i(t) < s_i(t) \implies \sum_{i=1}^{N} z_i(t) < 1.$$

This contradicts the assumption that $t \in (0, \tau)$, because the above condition requires that the auction terminates by period t at the latest.

Second, suppose t is such that $d_i(t') = 0$ for some $t' < t$, then $q_i(t) = q_i^-(t) = q_i(t')$ and $p_i(t) = p_i(t')$. Thus, $d_i(t) = 0$ and

$$z_i(t) = q_i(t) < s_i(t) \implies \sum_{i=1}^{N} z_i(t) < 1.$$

This also contradicts the assumption that $t \in (0, \tau)$.

Part 4. Suppose that $\tau > 1$. Then, for any time $t \in (1, \tau)$, $\sum_{i=1}^{N} z_i(t) > 1$. Yet, $d_i(t) = 0$ $\forall i, t > 1$. Thus, $q_i(1) = z_i(t) = q_i^-(t) \ \forall t \in (1, \tau)$. Let M be the set of all bidders i such that $q_i(1) > 0$. If $i \in M$, then

$$z_i(t) = q_i(t) = s_i(t) = 1 - \sum_{j \neq i} z_j(t) > 0, \ \forall t \in (1, \tau).$$

Thus, $1 = \sum_{i=1}^{N} z_i(t)$, which contradicts that $\sum_{i=1}^{N} z_i(t) > 1 \ \forall t \in (1, \tau)$.

Part 5. I prove this by contradiction. Suppose that $d_i(t) = 0$ for some $t \in (0, \tau)$, yet $\mathcal{Q}_i > 0$. Note that $d_i(t) = 0$ implies $q_i(t') = q_i(t)$, $p_i(t') = p_i(t)$, and $d_i(t') = d_i(t) = 0 \ \forall \tau > t' > t$. Thus, $\mathcal{Q}_i > 0 \implies \mathcal{Q}_i = z_i(t') = q_i(t') = q_i(t) > 0 \ \forall t' \in (t, \tau)$. Thus,

$$z_i(t') = q_i(t') = 1 - \sum_{j \neq i} z_j(t') \ \forall t' \in (t, \tau) \implies \sum_{i=1}^{N} z_i(t') = 1 \ \forall t' \in (t, \tau).$$

Thus, the auction terminates by time t' or earlier ($\tau \leq t'$). This contradicts our assumption that the auction ends at time $\tau > t'$.

Part 6. I prove this by contradiction. First, I show that $p_i(t) < b_i \; \forall t \in (0, \tau)$. Suppose that $t \in (0, \tau)$, yet $p_i(t) \geq b_i$ for some i. Then $d_i(t) = 0$. We have already shown that if $d_i(t) = 0$ for some $t < \tau$, then $\mathcal{Q}_i = \mathcal{P}_i = 0$. However, $\mathcal{P}_i \geq p_i(t) \geq b_i > 0$, where the second inequality follows because $\mathcal{P}_i \geq p_i(t) \; \forall t \in (0, \tau)$. Thus, $0 = \mathcal{P}_i > 0$ and we have a contradiction.

Next, I show $\mathcal{P}_i \leq b_i$. Note that $p_i^-(t) + td_i(t) \leq b_i \; \forall t \in (0, \tau)$. This holds because $p_i^-(t) = q_i^-(t)t - \int_0^t q_i(s)ds$ and $d_i(t) \leq \frac{b_i - p_i^-(t)}{t}$. Thus $p_i^-(t) + td_i(t) \leq b_i - p_i^-(t) + p_i^-(t) = b_i$. Finally, the construction of the proxy clinching auction implies, $\mathcal{P}_i \leq \lim_{t \to^- \tau} p_i^-(t) + td_i(t) \leq b_i$.

Proof of Proposition 1. I use an h superscript for variables when bidder i reports (v, b), where $v > v_i$; and an ℓ superscript when bidder i reports $(v_i, \min\{v_i, b\})$.

For any $t \leq v_i$, $z_i^\ell(t) = z_i^h(t) = \min\{1, q_i^-(t) + \frac{b - p_i^-(t)}{t}\} = \min\{1, \frac{b + \int_0^t q_i^\ell(s)ds}{t}\}$. Thus, $s_j^h(t) = s_j^\ell(t) \; \forall t \leq v_i$. Therefore, if $\tau^\ell < v_i$, then $\tau^h = \tau^\ell$ and bidder i receives an equal payoff in each case, $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h)$.

If $\tau^\ell = v_i$, then $\tau^h \geq v_i$. First, suppose $\tau^h = v_i$. Thus, $\mathcal{Q}_i^\ell \in [z_i^\ell(v_i), \lim_{t \to v_i^-} z_i^\ell(t)]$, and $\mathcal{Q}_i^h = z_i^h(v_i)$ because z_i^ℓ is left discontinuous at v_i and z_i^h is left continuous at v_i. In addition, $s_j^h(t) = s_j^\ell(t) \forall t \leq v_i$ implies that $q_i^h(t) = q_i^\ell(t) \; \forall t \in (0, v_i)$. Thus, $\mathcal{Q}_i^h = z_i^h(v_i) = \lim_{t \to v_i^-} z_i^h(v_i) = \lim_{t \to v_i^-} z_i^\ell(t) \geq \mathcal{Q}_i^\ell$, and $\mathcal{P}_i^h = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$. Thus, $\mathcal{P}_i^\ell > b_i$ only if $\mathcal{P}_i^h > b_i$. If $\mathcal{P}_i^\ell > b_i$, then $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = -\infty$. If $\mathcal{P}_i^h > b_i \geq \mathcal{P}_i^\ell$, then $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) > u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = -\infty$. Finally, if $b_i \geq \mathcal{P}_i^h \geq \mathcal{P}_i^\ell$, then

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = \mathcal{Q}_i^h v_i - \left(\mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)\right) = \mathcal{Q}_i^\ell v_i - \mathcal{P}_i^\ell = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Next, suppose that $\tau^\ell = v_i < \tau^h$. Since, $s_j^h(t) = s_j^\ell(t) \forall t \leq v_i$ and $\mathcal{Q}_i^\ell \leq s_j^\ell(v_i)$, then $q_i^h(v_i) \geq \mathcal{Q}_i^\ell$ and $p_i^h(v_i) = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$. In addition, $\mathcal{Q}_i^h \geq q_i^h(v_i) \geq \mathcal{Q}_i^\ell$ and $\mathcal{P}_i^h \geq p_i^h(v_i) + v_i(\mathcal{Q}_i^h - q_i^h(v_i)) = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$. Thus, $\mathcal{P}_i^h \geq \mathcal{P}_i^\ell$. If $\mathcal{P}_i^\ell > b_i$, then $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = -\infty$. If $\mathcal{P}_i^h > b_i \geq \mathcal{P}_i^\ell$, then $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) > u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = -\infty$. Finally, if $b_i \geq \mathcal{P}_i^h \geq \mathcal{P}_i^\ell$, then

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) \leq \mathcal{Q}_i^h v_i - \left(\mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)\right) = \mathcal{Q}_i^\ell v_i - \mathcal{P}_i^\ell = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Finally, suppose $\tau^\ell, \tau^h > v_i$. Then, $\exists t \in (0, \tau^\ell)$ such that $d_i^\ell(t) = 0$. Thus, $\mathcal{Q}_i^\ell = \mathcal{P}_i^\ell = 0$ by Lemma 1 and $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = 0$. Also, recall that $q_i^\ell(t) = q_i^h(t) = 0 \; \forall t < v_i$. Thus,

$\mathcal{P}_i^h = \mathcal{Q}_i^h \tau^h - \int_0^{\tau^h} q_i^h(s)ds = \mathcal{Q}_i^h \tau^h - \int_{v_i}^{\tau^h} q_i^h(s)ds \geq \mathcal{Q}_i^h v_i$. Thus,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) \leq \mathcal{Q}_i^h v_i - \mathcal{P}_i^h \leq \mathcal{Q}_i^h v_i - \mathcal{Q}_i^h v_i = 0 = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Proof of Lemma 2. I use the superscript h to denote variables when bidder i reports (v, b_h), and superscript ℓ for variables when bidder i reports type (v, b_ℓ), where $b_h > b_\ell$. I use notation $z_{-i}(t) := \sum_{j \neq i} z_j(t)$, and similarly, $q_{-i}(t) = \sum_{j \neq i} q_j(t)$.

The proof is by contradiction. Suppose there exists a time $t \in (0, \min\{\tau^h, \tau^\ell\})$ where

$$q_i^\ell(t) > q_i^h(t) \geq 0.$$

Then

$$q_i^h(t) = 1 - z_{-i}^\ell(t) > \max\{0, 1 - z_{-i}^h(t)\} = q_i^\ell(t) \implies z_{-i}^h(t) > z_{-i}^\ell(t).$$

Note that $z_{-i}^\ell(t) < 1$. Thus, $z_j^\ell(t) < 1 \; \forall j \neq i$. Then,

$$\sum_{j \neq i} \frac{b_j + \int_0^t q_j^h(s)ds}{t} \mathbb{I}_{v_j < t} \geq z_{-i}^h(t) > z_{-i}^\ell(t) = \sum_{j \neq i} \frac{b_j + \int_0^t q_j^\ell(s)ds}{t} \mathbb{I}_{v_j < t}.$$

Thus,

$$\sum_{j \neq i} \int_0^t q_j^h(s) \mathbb{I}_{v_j < t} ds > \sum_{j \neq i} \int_0^t q_j^h(s) \mathbb{I}_{v_j < t} ds.$$

Note, $q_j^h(s) > 0$ only if $s < v_j$, by Lemma 1. Thus, we can rewrite the above condition as

$$\int_0^t \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds > 0.$$

That is,

$$q_i^\ell(t) - q_i^h(t) > 0 \implies \int_0^t \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds > 0.$$

Let $\tilde{t}$ be defined as the infimum of all times where the above integral condition holds.

$$\tilde{t} := \inf\{t | \int_0^t \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds > 0\}.$$

Thus, $\forall t \leq \tilde{t}$, $q_i^h(t) \geq q_i^\ell(t)$. Note also that $\int_0^t \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds$ is continuous in t. Thus, $\int_0^{\tilde{t}} \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds = 0$. Combining the definition of $\tilde{t}$ with continuity, we get $\forall \epsilon > 0$, $\exists \delta > 0$, such that $t' \in (\tilde{t}, \tilde{t} + \delta) \implies \int_0^{\tilde{t}} \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds = 0 < \int_0^{t'} \left(q_{-i}^h(s) - q_{-i}^\ell(s)\right) ds < \epsilon$. This implies $\exists j$ such that,

$$q_j^h(t') > q_j^\ell(t') \geq 0.$$

Let M be the set of all bidders $j \neq i$ such that $q_j^h(t') > q_j^\ell(t')$. Then, if $j \in M$,

$$q_j^h(t') = 1 - z_{-j}^h(t') > \max\{0, 1 - z_{-j}^\ell(t')\} = q_j^\ell(t').$$

This implies, $z_{-j}^\ell(t') - z_{-j}^h(t') > 0$. Or equivalently,

$$z_{-j}^\ell(t') - z_{-j}^h(t') = \sum_{n \neq j} \frac{b_n^\ell + \int_0^{t'} q_n^\ell(s)ds}{t'} \mathbb{I}_{v_n < t'} - \sum_{n \neq j} \frac{b_n^h + \int_0^{t'} q_n^h(s)ds}{t'} \mathbb{I}_{v_n < t'} > 0.$$

$$\implies \int_0^{t'} \left(q_{-j}^\ell(s) - q_{-j}^h(s)\right) ds > b_h - b_\ell.$$

Thus,

$$\sum_{j \in M} \int_0^{t'} \left(q_{-j}^\ell(s) - q_{-j}^h(s)\right) > \#M \left(b_h - b_\ell\right).$$

We can rewrite the lefthand side as,

$$(\#M - 1) \sum_{j=1}^{N} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds + \sum_{j \notin M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds.$$

Looking at the first term, $\forall \epsilon > 0$,

$$(\#M - 1) \sum_{j=1}^{N} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds \leq \epsilon$$

when t' is sufficiently close to $\tilde{t}$, because $\int_0^{\tilde{t}} \left(q_{-i}^\ell(s) - q_{-i}^h(s)\right) = 0$ and $\int_0^{\tilde{t}} \left(q_i^\ell(s) - q_i^h(s)\right) \leq 0$ (because $q_i^h(t) \geq q_i^\ell(t)\ \forall t \leq \tilde{t}$). For the second term, I show

$$\sum_{j \notin M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds < -\#M \left((b_h - b_\ell) - \epsilon\right) < 0.$$

This holds because when t' is sufficiently close to $\tilde{t}$, then

$$\int_0^{\tilde{t}} \left(q_{-i}^\ell(s) - q_{-i}^h(s)\right) = 0 \approx \sum_{j \notin M, i} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds + \sum_{j \in M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s)\right) ds$$

Recalling that $\sum_{j \in M} \int_0^{t'} \left(q_{-j}^\ell(s) - q_{-j}^h(s)\right) > \#M \left(b_h - b_\ell\right)$, we have that $\forall \epsilon > 0$, t' sufficient close to $\tilde{t}$, implies

$$\sum_{j \notin M, i} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds < -\left(\#M \left(b_h - b_\ell \right) - \epsilon \right)$$

In addition, recall $\int_0^{\tilde{t}} \left(q_i^\ell(s) - q_i^h(s) \right) \leq 0$ because $q_i^h(t) \geq q_i^\ell(t)$ $\forall t \leq \tilde{t}$. Thus, when t' is sufficiently close to $\tilde{t}$,

$$\sum_{j \notin M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds < -\left(\#M \left(b_h - b_\ell \right) - \epsilon \right).$$

Thus when t' is sufficiently close to $\tilde{t}$,

$$(\#M - 1) \sum_{j=1}^{N} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds + \sum_{j \notin M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds < 0,$$

yet

$$(\#M - 1) \sum_{j=1}^{N} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds + \sum_{j \notin M} \int_0^{t'} \left(q_j^\ell(s) - q_j^h(s) \right) ds = \sum_{j \in M} \int_0^{t'} \left(q_{-j}^\ell(s) - q_{-j}^h(s) \right),$$

and

$$\sum_{j \in M} \int_0^{t'} \left(q_{-j}^\ell(s) - q_{-j}^h(s) \right) > \#M \left(b_h - b_\ell \right) > 0.$$

Proof of Lemma 3. I use the superscript h to denote variables when bidder i reports $(v, \min\{v, b_i\}) \in \Theta$, and superscript ℓ for variables when bidder i reports type $(v, b_\ell) \in \Theta$. , We assume $b_h := \min\{v, b_i\} > b_\ell$. I consider three cases.

Case 1: $\tau^\ell \geq \tau^h$ and $\tau^h \neq v$.

Note that z_i^h is continuous at $t = \tau^h$, as

$$z_i^h(t) = \begin{cases} \min\{1, \frac{b_h + \int_0^t q_i^h(s) ds}{t}\} & if\ t < v \\ 0 & if\ t \geq v. \end{cases}$$

Thus, $\mathcal{Q}_i^h = z_i^h(\tau^h) \geq z_i^\ell(\tau^h) \geq \mathcal{Q}_i^\ell$ where the first inequality holds because $\int_0^{\tau^h} q_i^h(s) ds \geq \int_0^{\tau^h} q_i^\ell(s) ds$ and the second holds because $z_i^\ell(t)$ is declining in t and $\tau^\ell \geq \tau^h$. Note that

$$\mathcal{P}_i^\ell = p_i^\ell(\tau^h) + \left(\mathcal{P}_i^\ell - p_i^\ell(\tau^h) \right) = \tau^h \mathcal{Q}_i^\ell - \int_0^{\tau^h} q_i^\ell(s) ds - \int_{\tau^h}^{\tau^\ell} q_i^\ell(s) ds.$$

Since $\tau^\ell \geq \tau^h$, then

$$\mathcal{P}_i^\ell \geq \mathcal{Q}_i^\ell \tau^h - \int_0^{\tau^h} q_i^\ell(s)ds.$$

Thus, if $\tau^h \leq v_i$,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) - u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) \geq \left(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell\right)\left(v_i - \tau^h\right) + \int_0^{\tau^h} (q_i^h(s) - q_i^\ell(s))ds \geq 0.$$

If $\tau^h > v_i$, then $\mathcal{Q}_i^h = \mathcal{Q}_i^\ell = \mathcal{P}_i^h = \mathcal{P}_i^\ell = 0$ because $d_i^h(t) = d_i^\ell(t) = 0$ when $t \in (v_i, \tau^h)$.

Case 2: $\tau^\ell \geq \tau^h = v$**.**

If $\tau^\ell > v$, then $\exists t$ such that $d_i^\ell(t) = 0$. Lemma 1 shows that this implies $\mathcal{Q}_i^\ell = \mathcal{P}_i^\ell = 0$. In addition, $\mathcal{P}_i^h \leq \tau^h \mathcal{Q}_i^h \leq v_i \mathcal{Q}_i^h$. Therefore,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = \mathcal{Q}_i^h v_i - \mathcal{P}_i^h \geq 0 = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

If $\tau^h = \tau^\ell = v$, then $z_i^n(t)$ is discontinuous at $t = \tau^h = \tau^\ell := \tau$. For all $t < \tau$, we have that $z_i^h(t) \geq z_i^\ell(t)$ because $z_i^h(t) = \min\{1, \frac{b_i + \int_0^t q_i^h(s)ds}{t}\} \geq \min\{1, \frac{b + \int_0^t q_i^\ell(s)ds}{t}\}$ as $b > b_i$ and $q_i^h(t) \geq q_i^\ell(t)$. At time τ, $z_i^h(\tau) = q_i^{-h}(\tau) \geq q_i^{-\ell}(\tau)$. Thus, $\mathcal{Q}_i^h \geq \mathcal{Q}_i^\ell$ by construction. In addition,

$$\mathcal{P}_i^h = \tau \mathcal{Q}_i^h - \int_0^\tau q_i^h(s)ds \leq \mathcal{P}_i^\ell + \tau(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell) = \tau \mathcal{Q}_i^h - \int_0^\tau q_i^\ell(s)ds,$$

since $\int_0^\tau q_i^h(s)ds \geq \int_0^\tau q_i^\ell(s)ds$. Thus, $v_i \geq v = \tau$ implies,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = \mathcal{Q}_i^h v_i - \mathcal{P}_i^h \geq \mathcal{Q}_i^h v_i - \left(\mathcal{P}_i^\ell + v_i\left(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell\right)\right) = u_i(\mathcal{Q}_i^\ell, \mathcal{P}_i^\ell).$$

Case 3: $\tau^h > \tau^\ell$**.**

If $\tau^\ell > v$, then $\exists t < \tau^\ell < \tau^h$ such that $d_i^h(t) = d_i^\ell(t) = 0$. Lemma 1 shows that this implies $\mathcal{Q}_i^\ell = \mathcal{Q}_i^h = \mathcal{P}_i^\ell = \mathcal{P}_i^h = 0$.

For the case where $v \geq \tau^\ell$, note that $z_i^h(t) \geq z_i^\ell(t)\ \forall t \in (0, \tau^\ell]$. Thus, at time τ^ℓ we have that

$$\sum_{i=1}^N z_i^h(\tau^\ell) > 1 \geq \sum_{i=1}^N z_i^\ell(\tau^\ell).$$

Recall, $q_i^{-h}(t) \geq q_i^{-\ell}(t) \geq\ \forall t \in [0, \tau^\ell]$. This implies,

$$s_i^h(\tau^\ell) \geq s_i^\ell(\tau^\ell) > 0.$$

Thus,

$$q_i^h(\tau) = \max\{0, 1 - \sum_{j \neq i} z_i^h(\tau)\}.$$

Note that $\mathcal{Q}_i^\ell \leq \max\{0, 1 - \sum_{j\neq i} z_j(\tau^\ell)\}$. Thus, $q_i^h(\tau^\ell) \geq \mathcal{Q}_i^\ell$. Recall that $\mathcal{P}_i^\ell = \tau^\ell \mathcal{Q}_i^\ell - \int_0^{\tau^\ell} q_i^\ell(s)ds$ and $p_i^h(\tau^\ell) = \tau^\ell q_i^h(\tau^\ell) - \int_0^{\tau^\ell} q_i^h(s)ds$. First suppose that $v' \geq \tau^\ell$. Then,

$$u_i(q_i^h(\tau^\ell), -p_i^h(\tau^\ell)) - u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = \left(q_i^h(\tau^\ell) - \mathcal{Q}_i^\ell\right)(v_i - \tau^\ell) + \int_0^{\tau^\ell} \left(q_i^h(s) - q_i^\ell(s)\right) ds \geq 0.$$

Note that $\mathcal{Q}_i^h \geq q_i^h(\tau^\ell)$ and $\mathcal{P}_i^h \leq p_i^h(\tau^\ell) + v(\mathcal{Q}_i^h - q_i^h(\tau^\ell)) \leq p_i^h(\tau^\ell) + v_i(\mathcal{Q}_i^h - q_i^h(\tau^\ell))$. Thus,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) \geq u_i(q_i^h(\tau^\ell), -p_i^h(\tau^\ell)) \geq u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Proof of Lemma 4. I use the superscript h to denote variables when bidder i reports $(v_i, b_i) \in \Theta$, and superscript ℓ for variables when bidder i reports type $(v, \min\{b_i, v\}) \in \Theta$. We assume $v < v_i$. First, I show if $\tau^\ell < v_i$, then $\tau^h = \tau^\ell$. To do this, I show that $z_i^h(t) = z_i^\ell(t)\ \forall t < v$. If $v < b_i$, then

$$z_i^\ell(t) = \max\{\frac{v + \int_0^t q_i^\ell(s)ds}{t}, 1\} = 1 = \max\{\frac{b_i + \int_0^t q_i^h(s)ds}{t}, 1\} = z_i^h(t)\ if\ t < v.$$

In addition, if $v \geq b_i$, then

$$z_i^\ell(t) = \max\{\frac{b_i + \int_0^t q_i^\ell(s)ds}{t}, 1\}\ if\ t < v.$$

This is the same function form as $z_i^h(t)$ if $t < v < v_i$. Thus there is no difference in bidder i reported preferences by reporting (v, b_i) or (v_i, b_i) when the time is $t < v$. Thus,

$$\sum_{j=1}^N z_j^h(t) = \sum_{j=1}^N z_j^\ell(t)\ if\ t < v.$$

This implies, $\tau^h = \tau^\ell = \inf\{t : \sum_{j=1}^N z_j^h(t) \leq 1\}$.

Next, I show that, $\tau^\ell \geq v \implies \tau^h \geq v$. This is because $z_j^h(t) = z_j^\ell(t)\ \forall j = 1, \dots, N$ and $t < v$. Thus, $\tau^\ell \geq v \implies \sum_{i=1}^N z_i^h(t) = \sum_{i=1}^N z_i^\ell(t) > 1\ \forall t < v$. Or equivalently $\tau^h \geq v$.

This allows me to break the remainder of the proof into three cases.

Case 1 $\tau^\ell < v$. There is no difference in the outcome of the auction prior to time v because, $z_i^h(t) = z_i^\ell(t)\ \forall t < v$. Since the auction terminates at time $\tau^\ell < v$ under either report, then $\mathcal{Q}_i^h = \mathcal{Q}_i^\ell$ and $\mathcal{P}_i^h = \mathcal{P}_i^\ell$.

Case 2 $\tau^\ell = v$**,** $\tau^h \leq v_i$**.** When bidder i reports the lower valuation, $z_i^\ell(t)$ is discontinuous at τ^ℓ. Thus, she wins $\mathcal{Q}_i^\ell \in [z_i^\ell(\tau^\ell), \lim_{t \to^- \tau^\ell} z_i^\ell(t)]$ units and pays $\mathcal{P}_i^\ell = \mathcal{Q}_i \tau^\ell - \int_0^{\tau^\ell} q_i^\ell(s)ds = \mathcal{P}_i^\ell = \mathcal{Q}_i \tau^\ell - \int_0^{\tau^\ell} q_i^h(s)ds$, where the final equality holds as $q_i^h(s) = q_i^\ell(s)\ \forall s \leq \tau^\ell$.

If $\tau^h = \tau^\ell = v$, then $\mathcal{Q}_i^h = z_i^h(v)$, because z_i^h is continuous at v. Thus, $\mathcal{Q}_i^h \geq \mathcal{Q}_i^\ell$ because $z_i^h(t) = \lim_{t \to^- v} z_i^h(t) = \lim_{t \to^- v} z_i^\ell(t) \geq \mathcal{Q}_i^\ell$ where the final inequality follows from the construction of the mechanism. In addition $\mathcal{P}_i^h = \mathcal{P}_i^\ell + v(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell) < \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$.

If $\tau^h > \tau^\ell = v$, then I first show $q_i^h(\tau^\ell) \geq \mathcal{Q}_i^\ell$. This holds because at time τ^ℓ we have that

$$\sum_{i=1}^{N} z_i^h(\tau^\ell) > 1 \geq \sum_{i=1}^{N} z_i^\ell(\tau^\ell).$$

Recall, $z_i^h(t) = z_i^\ell(t)\ \forall t \in [0, \tau^\ell]$. This implies,

$$q_i^h(\tau^\ell) = s_i^h(\tau^\ell) = s_i^\ell(\tau^\ell) > 0.$$

Note that $\mathcal{Q}_i^\ell \leq s_i^\ell(\tau^\ell) = \max\{0, 1 - \sum_{j \neq i} z_j(\tau^\ell)\}$. Thus,

$$\mathcal{Q}_i^\ell \leq s_i^\ell(\tau^\ell) = s_i^h(\tau^\ell) = q_i^h(\tau^\ell).$$

This implies $\mathcal{Q}_i^h \geq q_i^h(\tau^\ell) \geq \mathcal{Q}_i^\ell$, and

$$\mathcal{P}_i^h \leq \mathcal{P}_i^\ell + v(q_i^h(\tau^\ell) - \mathcal{Q}_i^h) + \tau^h(\mathcal{Q}_i^h - q_i^h(\tau^\ell)) \leq \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell).$$

Thus, when $\tau^\ell = v,\ \tau^h \leq v_i$, then $\mathcal{Q}_i^h \geq \mathcal{Q}_i^\ell$, and

$$\mathcal{P}_i^h \leq \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell).$$

Note also that $\mathcal{P}_i^h, \mathcal{P}_i^\ell \leq b_i$ by Lemma 1. Thus,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) = \mathcal{Q}_i^h v_i - \mathcal{P}_i^h \geq \mathcal{Q}_i^\ell v_i - \mathcal{P}_i^\ell = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Case 3. $\tau^\ell > v_i$**.** Since $\tau^\ell > v$, $\exists t \in (0, \tau^\ell)$ such that $d_i^\ell(t) = 0$. Lemma 1 then implies that $\mathcal{Q}_i^\ell = \mathcal{P}_i^\ell = 0$. If bidder i reports the high type, then she wins $\mathcal{Q}_i^h$ units and pays $\mathcal{P}_i^h \leq \mathcal{Q}_i^h v_i$, where the inequality follows from the construction of the auction. Since Lemma 1 implies $\mathcal{P}_i^h \leq b_i$, then

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) \geq 0 = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Proof of Proposition 3. We assume bidder i's opponents play undominated strategies. That is bidders $j \neq i$ play strategy profile $a_{-i} = (a_1 \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, where $a_j : \Theta \to$

$\Delta(\Theta)$, and $(v,b) \in supp\,(a_j(\theta_j))$ implies $0 \leq b_j \leq b \leq v \leq v_j \leq 1$.

Let $\mathcal{U}(\theta_i)$ be the set of undominated reports conditional on having type θ_i. Our prior results show that $(v,b) \in \mathcal{U}(\theta_i)$ only if either $0 \leq b_i < b \leq v \leq v_i \leq 1$, or $b_i = b$ and $v = v_i$. I want to show that

$$(v_i, b_i) \in \arg \max_{(v,b)\in\mathcal{U}(\theta_i)} \mathbb{E}_{F_i,a_{-i}}\left(U_i(\mathcal{Q}_i, -\mathcal{P}_i)|\theta_i\right),$$

and if $(v', b') \in \mathcal{U}(\theta_i)$ and $(v', b') \neq (v_i, b_i)$, then

$$(v', b') \notin \arg \max_{(v,b)\in\mathcal{U}(\theta_i)} \mathbb{E}_{F_i,a_{-i}}\left(U_i(\mathcal{Q}_i, -\mathcal{P}_i)|\theta_i\right).$$

First, note that the payoff from truthful reporting is non-negative. If bidder i reports her type truthfully, $\mathcal{P}_i \leq b_i$ and $\mathcal{P}_i \leq \mathcal{Q}_i v_i$. Thus

$$\max_{(v,b)\in\mathcal{U}(\theta_i)} \mathbb{E}_{F_i,a_{-i}}\left(U_i(\mathcal{Q}_i, -\mathcal{P}_i)|\theta_i\right) \geq 0 \ \forall F_i.$$

Second, note that for any realization of the proxy-clinching auction, bidder i's utility is bounded by $v_i \leq 1$ for any report (v,b). This is because $\mathcal{Q}_i v_i \leq 1$ and $\mathcal{P}_i \geq 0$.

If bidder i reports $(v,b) \neq (v_i, b_i)$, then $(v,b) \in \mathcal{U}(\theta_i) \implies 0 \leq b_i < b \leq v \leq v_i \leq 1$. Let $T_{-i} = \{\theta_{-i}|b > \max_{j\neq i} v_j$ and $\exists j$ $s.t.$ $b_j > b_i\}$. Note that this set has positive measure, $\mu(T_{-i}) > 0$. Full-support beliefs imply that, bidder i believes there is a positive probability that $\theta_{-i} \in T_{-i}$. That is, $F_i(T_{-i}|\theta_i) > 0$. If $\theta_{-i} \in T_{-i}$, then $a_{-i}(\theta_{-i})$ is such that all reported values v'_j have $\min\{b_j, v_j\} \leq v'_j \leq \max_{j\neq i} v_j < \delta$.

Thus, $\sum_{j\neq i} z_j(b_i + \epsilon) \geq 1$ where $\epsilon > 0$ is sufficiently small, because $\exists j$ with type θ_j such that $v_j, b_j > b_i$ and $z_j(b_i + \epsilon) = 1$. Thus $\tau > b_i + \epsilon$, and $q_i(b_i + \epsilon) = 0$. Yet, $\tau \leq \max_{j\neq i} v_j$, because if $\tau > \max_{j\neq i} v_j$, then $\exists\ t \in (\max_{j\neq i} v_j, \tau)$ $s.t.$ $\sum_{j\neq i} z_j(t) = 0$. Since $z_i(t) = 1$ $\forall t \leq \tau \leq \max_{j\neq i} v_j$, then $\mathcal{Q}_i = 1$. In addition $q_i(t) = 0$ when $t = b_i + \epsilon$. Thus $\mathcal{P}_i > b_i + \epsilon$.

Thus bidder i will receive a payoff $-\infty$ payoff if she reports (v,b) and $\theta_{-i} \in T_{-i}$. Since this occurs with positive probability and the payoff from participating in the auction is bounded at one, we find that bidder i receives a negative expected payoff from reporting $(v,b) \neq (v_i, b_i)$. Thus reporting (v,b) is never a best reply, and reporting (v_i, b_i) gives a greater expected payoff than any undominated strategy and is a unique best reply.

Proof of Proposition 4. Consider an endowment economy with two commodities and $N+1$ agents. The two commodities are the good and money. Agent 0 (the auctioneer) has utility over units of the commodity q and money m,

$$U_0(q,m) = m.$$

Agent 0 is endowed with 0 units of the commodity and $\sum \mathcal{P}_i$ units of money.

Agent $i \in 1, \ldots, N$ has preferences

$$U_i(q, m) = \begin{cases} v_i + m - b_i & if\ q > 1, \\ qv_i + m - b_i & if\ q \in [0, 1] \end{cases}.$$

Agent i is endowed with $\mathcal{Q}_i$ units of the good and $b_i - \mathcal{P}_i$ units of money.

This endowment economy has a Walrasian equilibrium where agents do not trade and the market clearing prices are τ for the good and 1 for money.

To show this, we must find each agent's Marshallian demands given her endowment when the price of the good is τ and the price of money is 1. This requires studying the outcome of the proxy clinching auction when bidders report types truthfully.

Consider a bidder i with valuation $v_i < \tau$. In the proxy clinching auction bidders reports her type truthfully, the auction ends at time τ. Thus, there exists a time $t \in (v_i, \tau)$ such that $d_i(t) = 0$. Thus, $\mathcal{Q}_i = \mathcal{P}_i = 0$ by Lemma 1. In the general equilibrium economy, this means that bidder i is endowed with 0 units of the good and b_i units of money. At prices τ (for the good) and 1 (for money), she consumes only money, and consumes b_i units. Thus, she does not trade any money.

Next, consider a bidder with valuation $v_i = \tau$. In the proxy clinching auction bidder i wins $\mathcal{Q}_i$ units and pays $\mathcal{P}_i = p_i^-(\tau) + \tau\left(\mathcal{Q}_i - q_i^-(\tau)\right) \leq b_i$. In the general equilibrium economy, this means that bidder i is endowed with $\mathcal{Q}_i$ units of the good and $b_i - \mathcal{P}_i$ units of money. When prices are $\tau = v_i$ and 1, she demands any combination of money m and the good q such that

$$m + \tau q = (b_i - \mathcal{P}_i) + \tau \mathcal{Q}_i.$$

Thus, she is indifferent between trading and not trading either commodity at these prices.

Finally, consider a bidder with valuation $v_i > \tau$. In the proxy clinching auctions, where all bidders report types truthfully, bidder i wins $\mathcal{Q}_i = z_i(\tau) = \min\{1, \frac{b_i + \int_0^\tau q_i(s)ds}{t}\}$ units, since $z_i(t)$ is continuous at $t = \tau$. She pays $\mathcal{P}_i \leq b_i$. Note that $\mathcal{P}_i = \tau\mathcal{Q}_i - \int_0^\tau q_i(s)ds$. Thus, $\mathcal{Q}_i = \min\{1, \frac{b_i + \int_0^\tau q_i(s)ds}{\tau}\} = \min\{1, \frac{\tau\mathcal{Q}_i + b_i - \mathcal{P}_i}{\tau}\}$.

In the general equilibrium economy, bidder i is endowed with $\mathcal{Q}_i$ units and $b_i - \mathcal{P}_i$ units of money. When $v_i > \tau$, she demands as much of the good that she can afford, up to a quantity of 1. Thus, she demands $\min\{1, \frac{\tau\mathcal{Q}_i + b_i - \mathcal{P}_i}{\tau}\}$ units of the good. Recalling that $\mathcal{Q}_i = \min\{1, \frac{\tau\mathcal{Q}_i + b_i - \mathcal{P}_i}{\tau}\}$, this means she demands $\mathcal{Q}_i$ units of the good. Her remaining wealth is spent to consume money. Thus, if $\mathcal{Q}_i < 1$, then $\mathcal{Q}_i = \min\{1, \frac{b_i + \tau\mathcal{Q}_i - \mathcal{P}_i}{\tau}\} = \frac{b_i + \tau\mathcal{Q}_i - \mathcal{P}_i}{\tau} \implies b_i = \mathcal{P}_i$. That is, agent i demands no money, and spends all of her budget on consuming the good. If $\mathcal{Q}_i = 1$, then agent i buys one unit of the good and, $\frac{\tau\mathcal{Q}_i + b_i - \mathcal{P}_i}{\tau} \geq 1 \implies b_i \geq \mathcal{P}_i$, and the bidder demands one unit of the good and $b_i - \mathcal{P}_i$ units of money.

Thus, there is a Walrasian equilibrium where the price of money is 1 and the price of the good is τ. The first welfare theorem implies that the Walrasian equilibrium is Pareto efficient. Thus, $\nexists\{q_i, m_i\}_{i=0}^N$ s.t. $\sum_{i=0}^N q_i \le 1$ and

$$U_i(\mathcal{Q}_i, b_i - \mathcal{P}_i) \ge U_i(q_i, m_i)\ \forall i = 1, \dots, N$$

and

$$U_0(0, \sum_{i=1}^N \mathcal{P}_i) = \sum \mathcal{P}_i \ge U_0(q_0, m_0) = m_0.$$

Noting that $u_0(q, m) = U_0(q, m)$ and $u_i(q, m) = U_i(q, m - b_i)\ \forall i = 1, \dots, N$, we have that the outcome of the proxy clinching is Pareto efficient.

Proof of Corollary 2. First, I show that reporting $(v_i, \min\{b, v_i\})$ weakly dominates reporting (v, b), when $v > v_i$. The proof is nearly identical to the proof of Proposition 1.

I use an h superscript for variables when bidder i reports (v, b), where $v > v_i$; and an ℓ superscript when bidder i reports $(v_i, \min\{b, v_i\})$. For any $t \le v_i$, $z_i^\ell(t) = z_i^h(t) = \min\{1, q_i^-(t) + \frac{b - p_i^-(t)}{t}\} = \min\{1, \frac{b + \int_0^t q_i^\ell(s)ds}{t}\}$. Thus, $s_j^h(t) = s_j^\ell(t)\ \forall t \le v_i$. Therefore, if $\tau^\ell < v_i$, then $\tau^h = \tau^\ell$ and bidder i receives an equal payoff in each case, $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h)$.

If $\tau^\ell = v_i$, then $\tau^h \ge v_i$. First, suppose $\tau^h = v_i$. Thus, $\mathcal{Q}_i^\ell \in [z_i^\ell(v_i), \lim_{t \to v_i^-} z_i^\ell(t)]$, and $\mathcal{Q}_i^h = z_i^h(v_i)$ because z_i^ℓ is left discontinuous at v_i and z_i^h is left continuous at v_i. In addition, $s_j^h(t) = s_j^\ell(t) \forall t \le v_i$ implies that $q_i^h(t) = q_i^\ell(t)\ \forall t \in (0, v_i)$. Thus, $\mathcal{Q}_i^h = z_i^h(v_i) = \lim_{t \to v_i^-} z_i^h(v_i) = \lim_{t \to v_i^-} z_i^\ell(t) \ge \mathcal{Q}_i^\ell$, and $\mathcal{P}_i^h = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell) \implies \mathcal{P}_i^h \ge \mathcal{P}_i^\ell$. Thus, if $\mathcal{P}_i^\ell \le b_i$, then

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = \mathcal{Q}_i^\ell v_i - \mathcal{P}_i^\ell = \mathcal{Q}_i^h v_i - \left(\mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)\right) \ge u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h).$$

If $\mathcal{P}_i^\ell > b_i$ then

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = \mathcal{Q}_i^\ell v_i - b_i - \varphi(\mathcal{P}_i^\ell - b_i) = \mathcal{Q}_i^h v_i - b_i - (\mathcal{P}_i^h - \mathcal{P}_i^\ell) - \varphi(\mathcal{P}_i^\ell - b_i) \ge \mathcal{Q}_i^h v_i - b_i - \varphi(\mathcal{P}_i^h - b_i),$$

and

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) \ge \mathcal{Q}_i^h v_i - b_i - \varphi(\mathcal{P}_i^h - b_i) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h).$$

Next, suppose that $\tau^\ell = v_i < \tau^h$. Since, $s_j^h(t) = s_j^\ell(t) \forall t \le v_i$ and $\mathcal{Q}_i^\ell \le s_j^\ell(v_i)$, then $q_i^h(v_i) \ge \mathcal{Q}_i^\ell$ and $p_i^h(v_i) = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$. In addition, $\mathcal{Q}_i^h \ge q_i^h(v_i) \ge \mathcal{Q}_i^\ell$ and $\mathcal{P}_i^h \ge p_i^h(v_i) + v_i(\mathcal{Q}_i^h - q_i^h(v_i)) = \mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)$. Thus, if $\mathcal{P}_i^\ell \le b_i$, then

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = \mathcal{Q}_i^\ell v_i - \mathcal{P}_i^\ell \ge \mathcal{Q}_i^h v_i - \left(\mathcal{P}_i^\ell + v_i(\mathcal{Q}_i^h - \mathcal{Q}_i^\ell)\right) \ge u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h).$$

If $\mathcal{P}_i^\ell > b_i$ then

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = \mathcal{Q}_i^\ell v_i - b_i - \varphi(\mathcal{P}_i^\ell - b_i) \ge \mathcal{Q}_i^h v_i - b_i - (\mathcal{P}_i^h - \mathcal{P}_i^\ell) - \varphi(\mathcal{P}_i^\ell - b_i) \ge \mathcal{Q}_i^h v_i - b_i - \varphi(\mathcal{P}_i^h - b_i),$$

and

$$u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) \geq \mathcal{Q}_i^h v_i - b_i - \varphi(\mathcal{P}_i^h - b_i) = u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h).$$

Finally, suppose $\tau^\ell, \tau^h > v_i$. Then, $\exists t \in (0, \tau^\ell)$ such that $d_i^\ell(t) = 0$. Thus, $\mathcal{Q}_i^\ell = \mathcal{P}_i^\ell = 0$ by Lemma 1 and $u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell) = 0$. Also, recall that $q_i^\ell(t) = q_i^h(t) = 0$ $\forall t < v_i$. Thus, $\mathcal{P}_i^h = \mathcal{Q}_i^h \tau^h - \int_0^{\tau^h} q_i^h(s) ds = \mathcal{Q}_i^h \tau^h - \int_{v_i}^{\tau^h} q_i^h(s) ds \geq \mathcal{Q}_i^h v_i$. Thus,

$$u_i(\mathcal{Q}_i^h, -\mathcal{P}_i^h) \leq \mathcal{Q}_i^h v_i - \mathcal{P}_i^h \leq \mathcal{Q}_i^h v_i - \mathcal{Q}_i^h v_i = 0 = u_i(\mathcal{Q}_i^\ell, -\mathcal{P}_i^\ell).$$

Thus, reporting (v_i, b) weakly dominates reporting $(v, \min\{b, v_i\})$, when $v > v_i$. In addition, we can invoke Proposition 2, because the proof of Proposition 2 does not change if φ is finite or infinity. Thus, if bidders have utility functions of the form

$$u(x, -p, \theta_i) = \begin{cases} xv - p & if \ p \leq b_i \\ xv - b_i - \varphi(p - b_i) & if \ p > b_i \end{cases},$$

then $\mathcal{U}(\theta_i)$ is such that $a \in \mathcal{U}(\theta_i)$ only if

$$a = (v, b) \text{ where } (v, b) = (v_i, b_i) \text{ or } b_i < b \leq v \leq v_i.$$

Proof of Proposition 5. Consider any report (v, b) where $0 \leq b_i < b \leq v \leq v_i$. Let $T_{-i} = \{\theta_{-i} | b > \max_{j \neq i} v_j \text{ and } \exists j \ s.t. \ b_j > b_i\}$. Note that this set has positive measure, $\mu(T_{-i}) > 0$. Full-support beliefs imply that bidder i believes there is a positive probability that $\theta_{-i} \in T_{-i}$. That is, $F_i(T_{-i}|\theta_i) > 0$. If $\theta_{-i} \in T_{-i}$, then $a_{-i}(\theta_{-i})$ is such that all reported values v_j' have $\min\{b_j, v_j\} \leq v_j' \leq \max_{j \neq i} v_j < \delta$.

Thus, $\sum_{j \neq i} z_j(b_i + \epsilon) \geq 1$ where $\epsilon > 0$ is sufficiently small, because $\exists j$ with type θ_j such that $v_j, b_j > b_i$ and $z_j(b_i + \epsilon) = 1$. Thus $\tau > b_i + \epsilon$, and $q_i(b_i + \epsilon) = 0$. Yet, $\tau \leq \max_{j \neq i} v_j$, because if $\tau > \max_{j \neq i} v_j$, then $\exists \ t \in (\max_{j \neq i} v_j, \tau) \ s.t. \ \sum_{j \neq i} z_j(t) = 0$. Since $z_i(t) = 1$ $\forall t \leq \tau \leq \max_{j \neq i} v_j$, then $\mathcal{Q}_i = 1$. In addition $q_i(t) = 0$ when $t = b_i + \epsilon$. Thus $\mathcal{P}_i > b_i + \epsilon$. This occurs with probability $F_i(T_{-i}|\theta_i)$.

Recall that bidder i's utility is bounded. The highest payoff bidder i can receive is when she wins 1 unit and pays 0. Thus, her utility is bounded by v_i. Thus, bidder i's expected utility from reporting type (v, b) is bounded by

$$F_i(T_{-i}|\theta_i) \left(v_i - b_i - \varphi(\epsilon)\right) + \left(1 - F_i(T_{-i}|\theta_i)\right) v_i.$$

The above quantity is less than 0 when φ is sufficiently large. Thus, reporting (v_i, b_i) gives strictly greater expected utility than reporting (v, b) where $0 \leq b_i < b \leq v \leq v_i$, when φ is sufficiently large.

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