Large Multi-Unit Auctions with a Large Bidder

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Abstract

We compare equilibrium bidding in uniform-price and discriminatory auctions when a single large bidder (i.e., with multi-unit demand) competes against many small bidders, each with single-unit demands. We show that the large bidder prefers the discriminatory auction over the uniform-price auction, and we provide general conditions under which small bidders have the reverse preference. We also show that the discriminatory auction provides greater incentives for the large bidder to invest in increased capacity, while the uniform price auction provides greater incentives for small bidders to enter the auction. We use examples to show that the efficiency and revenue rankings of the two auctions are ambiguous.

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1 Introduction

Multi-unit auctions are often used to sell many units of a homogeneous good in markets with many buyers. Prominent examples include the markets for treasury bills, Initial Public Offerings of stock, and carbon emissions permits. Most of the multi-unit auctions used in practice are variants of the uniform-price (UP) or the discriminatory-price (DP) auction.[1] While there is no explicit characterization of the equilibrium of either auction in a general setting, **??** shows that it is possible to characterize each equilibrium when the auction is large (i.e., it involves many bidders and objects) and all bidders demand a negligible amount of the total issuance. Asymptotically no bidder in Swinkels' model influences the market's clearing price (i.e., the lowest price at which goods are awarded) with her actions and hence has no market power. Swinkels uses this observation to show that all bidders, as well as the seller, are approximately indifferent between the two formats.

Yet the presence of market power is an important feature in many large auction settings, and furthermore the degree of market power is not uniformly distributed across bidders. For example, **?** analyze the difference in bid behavior between small and large bidders in the Texas electricity sport market. While many bidders compete for the right to sell electricity, the largest bidder controls 24% of distribution. There is evidence that bidders have market power in U.S. Treasury auctions as well. **?** report that primary dealers in U.S. Treasury auctions are allocated 46% to 76% of the competitive demand.

We take a first step in augmenting a large auctions model to allow for market power. In our model, a single large bidder demands a non-negligible measure of the total issuance and competes against a continuum of small bidders. Small bidders are heterogeneous and have negligible demand when considered as a fraction of the total issuance. All bidders have private values.[2] If we think of **??** as modeling perfect competition in a multi-unit auction, then we model a market akin to a monoposony.[3]

Although the revenue and efficiency of the UP and DP auctions are generally ambiguous in the model, we obtain clear predictions for the bidders' preferences between the two formats.[4] A straightforward argument establishes that the large bidder prefers the DP auction in this environment. Similar to the bidders in Swinkels' model, the small bidders' bids do not influence the clearing price in either format but do affect the price they pay in the DP auction. They consequently shade their bids below their values in the DP auction but not the UP auction. On the other hand, any serious bid by the large bidder determines the clearing price in both formats. Given the small

[1]The discriminatory-price auction is sometimes called the pay-as-bid auction. We describe the pricing rules in Section **??**.

[2]This modeling technique is similar in spirit to **?**. Their paper considers a search model where buyers are matched with sellers. There is a single large seller who competes against a continuum of small sellers. Similarly, **?** model a corporate takeover where a firm is owned by many small shareholders and one large shareholder. They show the presence of a large shareholder makes profitable takeovers more likely.

[3]There is a continuum of bidders in our model, but we use the term monopsony because only one bidder has market power, who can be understood as purchasing units from a residual supply curve.

[4]The finding that the revenue and efficiency rankings are ambiguous is consistent with other studies of finite multi-unit auction markets, including **?**.

bidders' strategies, for each quantity that the large bidder might win she must pay strictly more for that quantity in the UP auction than she would in the DP auction. Hence, the large bidder favors the DP auction. At the same time, we provide general conditions under which small bidders typically have the reverse preferences over auction formats.

Bidder preferences over pricing rules have important practical implications. An immediate implication is that the bidder preferences can influence the choice of auction format. **?** report that "Similarly, in the lumber tract auctions in the Pacific Northwest, the local 'insiders' with neighbouring tracts have forcefully (and successfully) lobbied for open auctions and the elimination of sealed high-bid auctions" (pg. 425). They suggest that this outcome was the result of a strong bidder's preference for a first-price auction over a second-price auction, where "strong" means that their distribution stochastically dominates their opponent's in the reversed hazard rate order.[5]

Our model suggests that when comparing the UP and DP formats in a multi-unit setting size, not strength, determines bidder preferences. The relative strength of the bidders' distributions is important in other contexts. In our analysis of the DP auction we show that determining equilibrium bid behavior can be reduced to characterizing bid behavior in an asymmetric first-price auction. We show that a similar connection exists between the Vickrey auction and the second price auction. This allows us to use the results of **?** to show that a large bidder prefers the Vickrey auction to the DP auction when she is sufficiently strong relative to her rivals.

Bidder preferences over auction formats also indirectly affect how bidders make ex-ante decisions in each auction. We show this by looking at models of pre-auction investment and entry. Our multi-unit environment allows us to study the large bidder's incentive to invest in increasing capacity — a concept with no direct analog in the single-unit setting. Investment in capacity is distinct the from models of ex-ante auction investment that have been developed in the single-unit environment. In the single-unit setting **?** studies a case where a single bidder invests ex-ante to (stochastically) reduce the cost of providing a procured service. They show that such an investment is less beneficial prior to a first-price auction than a second-price auction because the increased strength of the investor leads her competitors to bid more aggressively in the first-price auction. Thus, the connection between the DP auction and the asymmetric first price auction shows that Arozamena and Cantillon's result imply that when the large bidder makes a value-enhancing investment in the DP auction, her rivals bid more aggressively.

Our results on investment in capacity in the DP auction are qualitatively different than Arozamena and Cantillon's results on value-enhancing investment. While Arozamena and Cantillon's results show that the a value-enhancing investment by the large bidder's induces her rival's to bid more aggressively, we show that capacity expansion has a reverse indirect effect. An increase in the large bidder's capacity has an effect similar to a strengthening of the small bidder's distribution in an asymmetric first-price auction. We show that this leads the small bidders to bid less aggressively in the DP auction. While the large bidder is weakly better off with higher capacity in both the DP and UP auctions, this indirect benefit is absent in the UP auction since the bidding behavior of the

[5] We define this ordering in Definition **??**.

small bidders is not affected. We use this observation to show that the large bidder is more inclined to make costly capacity expanding investment in the DP auction than in the UP auction.

We then compare entry incentives in the two auctions by extending the model to allow for endogenous entry of small bidders. More small bidders enter the uniform-price auction, because all else equal, small bidders get a higher payoff in the UP auction than the DP auction. Consequently more small bidders enter in the former than in the latter. The entry effect therefore only strengthens the large bidder's preference for the DP auction over the UP auction.

As a robustness check, we extend the model to allow for aggregate demand uncertainty. In our base model, the large bidder's preference for the DP auction is stark. In each auction, she faces a continuum of small bidders with a known distribution. Thus, in the equilibrium of each auction, the large bidder has an informational advantage as she perfectly infers the clearing price that results from any bid given the small bidders' strategies. We relax this assumption by making the small bidders' distribution uncertain for the large bidder. Our model extension is similar in style to the model of **?**, except that we also include a large bidder.

We show that adding uncertainty (through a mean-preserving spread) around the distribution of small bidder values can leave the large bidder's payoff unchanged in the DP auction but increases her payoff in the UP auction. A common feature of equilibria in the DP auction is bidders submit bid curves that have flat (i.e., constant) sections (**?**). If the large bidder's bid is entirely flat, her payoff is determined solely by the mean of the distributions of the small bidders, and hence is unaffected by mean-preserving spreads. In contrast, the UP auction essentially allows the large bidder to costlessly choose a different clearing price for each realization of the aggregate uncertainty. This means that aggregate uncertainty increases the large bidder's payoff in the UP auction but not the DP auction. In fact, we construct an example that shows that introducing aggregated demand uncertainty can lead to a preference reversal for the large bidder. In the example, the large bidder's distribution is strong, and this reduces the small bidders' incentives to shade their bids in the DP auction, making the DP auction less attractive to the large bidder. When combined with aggregate demand uncertainty, we see the two effects lead to the large bidder getting a higher payoff in the UP auction.

Yet the model with aggregate uncertainty also illustrates the robustness of our original results. We show there is no preference reversal if the amount of aggregate uncertainty is small or if the large bidder's value distribution is not as strong as small bidders'. In addition, it is natural to assume that the large bidder faces relatively little aggregate demand uncertainty, and thus has an informational advantage. Recent empirical research on treasury auctions supports the assumption that large bidders have substantial informational advantages (**?**).

The remainder of the paper is organized as follows. Section **??** describes the main model. Section **??** gives the main results, while Section **??** studies an extension in which the distribution of small bidders is uncertain. Section **??** concludes.

2 Model

A continuum of infinitesimal bidders of measure μ_s, referred to as the small bidders, compete in an auction for one unit of a divisible good against a large bidder with demand for a non-zero measure, μ_L of the good. The small bidders are "single-unit bidders" each with value for an increment dq, denoted by v_S and distributed according to the commonly known absolutely continuous distribution function $F_S(v_S) : [0, 1] \to \mathbb{R}_+$. We assume that the density, $f_S(v_S)$, is strictly positive on its support.

The large bidder has multi-unit demand and a constant marginal value for additional units. Her willingness to pay for each marginal unit is given by her type, t. The quantity demanded by the large bidder is bounded above by her "capacity" μ_L. Thus, the large bidder gets zero marginal value from winning any quantity of units beyond μ_L. We assume throughout that $\mu_S + \mu_L > 1$ (i.e., that there is excess demand for all units) and $\mu_L \leq 1$. The large bidder's type is distributed according to the commonly known absolutely continuous distribution function $F_L(t) : [0, \bar{t}] \to \mathbb{R}_+$ with density $f_L(t)$, which we also assume to be strictly positive on its support.

The "flat demand" assumption allows us to use results from the literature on asymmetric first-price auctions directly in our analysis of the discriminatory-price auction. A couple of our results generalize easily to cases where the large bidder's demand is downward sloping, and we note these cases.[6]

We evaluate three pricing rules, the discriminatory-price rule (DP), the uniform-price rule (UP), and the Vickrey rule.[7] In all three auctions, a type-v_S small bidder submits a bid $b(v_S)$ and a type-t large bidder submits a nonincreasing function of q, $B(q, t)$. Given the large bidder's type and the functions b and B, the total quantity demanded at price p is

$$Q(p; b, B) = \int_{b(v_s) \geq p} f_S(v_S)\, dv_S + \inf\{q | B(q, t) \geq p\}.$$

The clearing price, p^*, is determined by

$$p^* \equiv \sup\{p' | Q(p'; b, B) \geq 1\}.$$

In the UP auction, goods are awarded to all bidders with bids above the clearing price. The payment for each increment is $p^* dq$. In the DP auction payments are determined by bids directly, so a winning small bidder of type v_S gets payoff $v_S - b(v_S)$. The large bidder gets payoff,

$$\int_{B(q,t) \geq p^*} (t - B(q, t))\, dq.$$

In the Vickrey auction, the large bidder pays the integral sum of the defeated small bidders' values. The winning small bidders pay the clearing price.

[6] We believe that our results would be qualitatively unchanged with downward sloping demand, but to prove this would require an extensive analysis of a game similar to but different from an asymmetric first-price auction.

[7] Our analysis focuses on the UP and DP auctions, but we include the Vickrey auction as an efficient benchmark that is useful for comparison.

In Section **??**, we extend this model to allow for uncertainty in the *distribution* of small bidders' values.

3 Main Results

3.1 Bidder preferences

? shows that bidders are indifferent between participating in a discriminatory and a uniform-price auction in a large auction setting where no bidder has market power. The result holds because in each auction all bidders face a decision problem that is asymptotically similar when they have negligible demand and they face many rivals, even if bidders are ex-ante asymmetric.

In this section we show that bidders generally have clear preference rankings over the DP and UP auctions in our model. As in Maskin and Riley's model, the bidder's preference is driven by ex-ante asymmetries. Yet, like **?** we show that ex-ante differences in bidder strength do not influence bidder rankings of the two auctions. Instead we find that a bidder's size — whether it be large or small — determines a bidder's preferences over the two auctions.

In order to determine bidder preferences over the two auctions, we first characterize equilibrium bid behavior. In the UP auction bid behavior is determined using two rounds of iterative elimination of dominated strategies. First, note that it is always a best response for a small bidder to bid truthfully. This is because the small bidder does not have any impact on the market clearing price, and thus treats it as being exogenous. By bidding truthfully, the small bidder wins if and only if the market clearing price is below her value.

Remark 1. *In the UP auction, bidding $b_S(v) = v$ is a weakly dominant strategy for a small bidder.*

Remark **??** converts the large bidder's decision problem into a standard monopsony pricing problem. If the large bidder's highest losing bid is B,[8] the large bidder wins the quantity determined by the residual supply curve at price B, $S(B) := \max\{0, 1-(1-F_S(B))\mu_s\}$.[9] Thus, the large bidder's payoff for a given highest losing bid B is

$$\Pi_U(B,t) = S(B)(t-B).$$

The optimal bid maximizes the above expression. It is without loss of generality to assume that the large bidder submits a flat bid curve in this case, as only the highest losing bid affects the outcome of the auction. Note that the optimal bid is nondecreasing in the large bidder's type.

The UP auction is inefficient and the large bidder buys too few units relative to the efficient benchmark. This is because the large bidder has a demand reduction incentive that is equivalent to a monopsonist's incentive to lower the market price below the perfectly competitive benchmark.

[8] We ignore complications in the choice of price that may arise from a discontinuous bid curve here because the equilibrium bid curve will be a constant function.

[9] Note that all bids above 1 are weakly dominated, as the large bidder may win all of the units for a price of 1.

Small bidders do not affect the clearing price, and hence do not react to the large bidder's shading. This is not true in the DP auction, which we analyze next.

In the DP auction, small bidders do not have a dominant strategy and shade their bids. The large bidder again determines her bid by solving an optimization problem akin to that of a price-setting monopsonist. However, the residual supply available to the large bidder depends on the strategies of small bidders. In equilibrium the small bidders who do not win with probability one bid according to an increasing bid strategy $b_S(v)$ with inverse $\phi_S(b)$. Small bidders who win with probability one in equilibrium will all place the same bid. Thus, for these bidders, there is no well-defined inverse. In equilibrium these bidders always bid higher than the large bidder and thus they can be ignored when evaluating the large bidder's objective. For the sake of exposition, we will talk about $\phi_S(b)$ as "the inverse bid function". The large bidder optimizes by considering the residual supply curve

$$S(\phi_S(B)) := \max\{0, 1 - (1 - F_S(\phi_S(B)))\mu_s\}.$$

In the equilibrium of the DP auction, the large bidder submits a flat demand curve. In the UP auction, any bid function submitted by the large bidder that leads to the same clearing price is payoff equivalent the auctioneer and all bidders. In the DP auction, this is no longer true. A bid function that is strictly decreasing over some regions is never a best reply to any bid strategy of the small bidders. The large bidder receives a strictly higher payoff by submitting a flat demand curve that equals the clearing price. By submitting a flat demand curve that equals the clearing price, the large bidder wins the same number of units, but pays a lower price per unit. This observation is important, because it allows us to parametrize the large bidder's bid function by a single dimensional variable — the value of her flat bid. Thus, given the small bidders strategy ϕ_S and the large bidders type t, the large bidder selects B to maximize Π_D where

$$\Pi_D(B, t; \phi_S) := S(\phi_S(B))(t - B).$$

Since bidding at or above their value is a dominated strategy for small bidders, $S(B) \geq S(\phi_S(B))$, which leads the large bidder to prefer the DP pricing rule in a strong sense (i.e., for all undominated strategies that may be used by the small bidders). Note that we could use the same argument to show that the large bidder prefers the DP auction, even in a setting where the large bidder has declining marginal value for additional units.

Proposition 1. *The large bidder gets a higher payoff from the DP auction than from the UP auction,*

$$\max_B \Pi_D(B, t; \phi_S) \geq \max_B \Pi_U(B, t) \;\forall \phi_S \;\; s.t. \;\; \phi_S(b) \geq b \;\forall b.$$

Proof. Suppose $\phi_S(b) \geq b \;\forall b$. Then, $S(\phi_S(B)) \geq S(B)$ because S is weakly increasing in B by construction. If $B^* \in \arg\max_B \Pi_U(B, t)$, clearly $B^* \leq t$, implying

$$\Pi_U(B^*, t) \leq S(\phi_S(B^*))(t - B^*) = \Pi_D(B^*, t; \phi_S) \implies \max_B \Pi_U(B, t) \leq \max_B \Pi_D(B, t; \phi_S), \;\forall t.$$

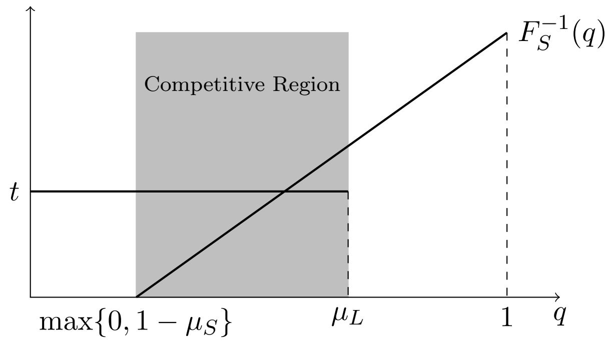

Figure 1: The Competitive Region

□

The above proposition shows that the large bidder prefers to participate in the DP auction over the UP auction, no matter what her type is. We draw deeper insights into bidder preferences by noting that the DP auction has strategic similarities to an asymmetric first-price auction between two bidders.

To see the connection between the DP auction and the asymmetric first-price auction, consider an equilibrium of the DP auction where the large bidder bids $B(t)$ and small bidders bid $b_S(v)$. In equilibrium, there is some interval (b_ℓ, b_h) on which bids are "competitive", meaning they win with probability strictly between zero and one. If a small bidder's value is such that her bid is in this open interval, then her bid wins if and only if her bid exceeds the bid of the large bidder. Similarly, the large bidder wins a quantity $q \in (\max\{0, 1-\mu_S\}, \mu_L)$ if and only if her bid is in the interval (b_ℓ, b_h). Thus, the large bidder directly competes with small bidders who bid in the competitive interval (b_ℓ, b_h). In addition, the large bidder's interim payoff is written identically to that of a bidder in an asymmetric first-price auction who bids against a rival that employs bid strategy $b_s(v)$ and has value that is distributed over $[\phi_S(b_\ell), \phi_S(b_h)]$, where ϕ_S is inverse bid function mentioned earlier. We denote the distribution of small bidder conditional on a small bidders type being in interval $[\phi_S(b_\ell), \phi_S(b_h)]$ as $\widetilde{F}_S(v)$. Note that this distribution,

$$\widetilde{F}_S(v) = \frac{1}{\mu_L} - \frac{\mu_S}{\mu_L}(1 - F_S(v)),$$

can be determined from the exogenous quantities μ_S and μ_L (see Figure **??**). We conclude that the equilibrium bid functions in the discriminatory auction are equivalent to the equilibrium bids in an asymmetric auction where one bidder has distribution F_L and the other has distribution $\widetilde{F}_S$.

Lemma 1. *An equilibrium in the DP auction can be constructed from the equilibrium of a first-price auction between two bidders with values distributed according to $\widetilde{F}_S(v)$ and $F_L(v)$. Small bidders who are sure winners (i.e., those with value $v > v_h$ s.t. $\widetilde{F}_s(v_h) = 1$) bid the largest equilibrium bid from the constructed first-price auction, while any sure losers (i.e., those with value $v < v_l$ s.t. $F_L(v_l) = 0$) bid their values.*

Proof. See Appendix. □

A similar construction connects the Vickrey auction to the equilibrium of a single-unit second-price auction. The logic is easier to see because it is an equilibrium to bid one's value on every marginal unit. Small bidders in the competitive region only win if their value exceeds the large bidder's, and the large bidder wins a quantity determined as the fraction of small bidders she defeats. This second lemma is useful when using analogies to the single-unit auctions to compare payoffs.

Lemma 2. *An equilibrium in the Vickrey auction can be constructed from a second-price auction between two bidders with values distributed according to* $\widetilde{F}_S(v)$ *and* $F_L(v)$*.*

From the single-unit auction constructions several corollaries follow. The first corollary establishes bidder preference between the Vickrey and UP auctions. Recall, that in the UP auction, the large bidder wins relatively fewer units than she would in the efficient outcome, and small bidders win relatively more frequently than they do in the efficient outcome.

Corollary 1. *The large bidder prefers the Vickrey auction to the UP auction. The small bidder prefers the UP auction to the Vickrey auction.*

In the literature on asymmetric first-price auctions, bidders are labeled "strong" or "weak" according to how their respective distributions are stochastically ordered. It is typical to use the reversed hazard rate order, $F \succeq_{rh} G$, defined as

Definition 1. $F \succeq_{rh} G \iff \frac{F(x)}{G(x)}$ *is nondecreasing for all* $x \in [0, \max\{1, \bar{t}\}]$.[10]

? show that in an asymmetric single-unit auction the weaker bidder prefers a first-price auction over a second-price auction, while the strong bidder has the reverse preference. Thus, Lemmas **??** and **??** imply that if the large bidder's distribution is strong relative to $\widetilde{F}_S(v)$ (i.e., $F_L \succeq_{rh} \widetilde{F}_S$), the large bidder prefers the Vickrey auction to the DP auction; and conversely if the large bidder's distribution is weak relative to $\widetilde{F}_S(v)$, the large bidder prefers the DP auction to the Vickrey auction. In each case, small bidders have the reverse preference between the two auctions.

Corollary 2. *If* $\widetilde{F}_S(v) \succeq_{rh} F_L(v)$*, then the large bidder prefers the DP auction to the Vickrey auction. If* $F_L(v) \succeq_{rh} \widetilde{F}_S(v)$*, then the large bidder prefers the Vickrey auction to the DP auction.*

If the small bidders prefer the Vickrey auction to the DP auction, which is the case when they are relatively strong, then a ranking between the UP auction and the DP auction follows immediately.

Corollary 3. *If* $\widetilde{F}_S(v) \succeq_{rh} F_L(v)$*, then the small bidders prefer the UP auction to the DP auction.*

Thus, we have provided conditions under which we can obtain a ranking of the DP, UP, and Vickrey auctions from the perspective of the large bidder. For the small bidder, we have shown that a UP auction is preferred to a Vickrey auction, and we have given conditions under which small bidders prefer the DP auction to the Vickrey auction.

Example **??** gives an illustration of the large bidder's ranking of the three auctions.

[10] When $f(x)$ and $g(x)$ both exist, this implies $f(x)/F(x) \geq g(x)/G(x)$.

Example 1. *Suppose that* $\mu_L = \mu_S = 1$ *and let*

$$F_L(t) = \left(\frac{t}{2\alpha}\right)^{\frac{\alpha}{1-\alpha}}$$

where $0 < \alpha < 1$ *and* $0 \leq t \leq 2\alpha$. *Also let the small bidder's values be uniformly distributed on* $[0, 1]$. *Let* B_i, Π^*_i *with* $i \in \{U, D, V\}$ *be respectively the equilibrium bid function and interim expected payoffs of the large bidder. Similarly, we let* $S_i(t)$ *be the total surplus generated by each format conditional on the large bidder's type. In equilibrium, we find that*

$$\begin{array}{lll} B_U(t,q) = \frac{t}{2} & \Pi^*_U(t) = \frac{t^2}{4} & S_U(t) = \frac{1}{2}\left(1 + \frac{3t^2}{4}\right) \\ B_D(t,q) = \frac{t}{2} & \Pi^*_D(t) = \frac{t^2}{4\alpha} & S_D(t) = \frac{1}{2}\left(1 + \frac{(4\alpha-1)t^2}{4\alpha^2}\right) \\ B_V(t,q) = t & \Pi^*_V(t) = \frac{t^2}{2} & S_V(t) = \frac{1}{2}\left(1 + t^2\right) \end{array}$$

and that the small bidders bid according to $b(v) = \alpha v$ *in the DP auction. For any* t *the revenue in the UP auction is* $B_U(t)$, *making the expected revenue* $\alpha/2$. *The expected revenue in the DP auction is greater at* $2\alpha/3$.

A difficulty arises in deciding the small bidders' ranking between the DP and UP auctions, and the small bidders' preferences over the two auction formats are important when we talk about the implications of auction choice on bidder entry in Section **??**. In Example **??** there is no ambiguity in the preference of small bidders — all small bidders receive a higher payoff in the UP auction than they do in the DP auction. This is because small bidders shade their bid in the latter, but not in the former. At the same time, the large bidder bids according to the same bid function in both auctions. Thus, a small bidder with type v wins with a higher probability in the UP auction $q_U(v)$ than the DP auction $q_D(v)$. If we let π_U and π_D be the interim expected payoff of the small bidder in the two auctions, then a standard envelope theorem argument shows that

$$q_U(v) \geq q_D(v) \forall v \implies \pi_U(v) = \int_0^v q_U(s)ds \geq \int_0^v q_D(s)ds = \pi_D(v).$$

While in Example **??** it is the case all types of small bidders have higher interim win probabilities in the UP auction than the DP auction, this does not generalize. In Example **??**, we show that when the distribution of small bidders who have values in the competitive region ($\widetilde{F}_S$) is sufficiently weak relative to the large bidder's, then some types of small bidders can have greater interim win probabilities in the DP auction versus the UP auction.

This possibility is easiest to see when μ_L is small. When μ_L is small in the UP auction and the large bidder is a high type (i.e., a type near $\bar{t}$), then the large bidder exhaust her capacity and reports a bid that ensures she wins μ_L units. However, similar to a first-price auction, it is only the highest type of large bidder that purchases the full amount μ_L in the DP auction. Any other type purchases less than the full amount. This implies that the higher types of small bidders must be outbidding more large bidder types in the DP auction when μ_L is small. Hence, they can have a higher interim win probability in the DP auction.

If some types of small bidders win with a higher interim win probability in the DP, then we can no longer uses the standard envelope theorem argument to show that all types of small bidders prefer the UP auction. However, in Example **??** we still see that all types of small bidders prefer the UP auction to the DP auction, even though higher types of small bidders win with greater probability in the later. We use Example **??** to motivate the construction of general conditions under which the $UP \geq DP$ ranking holds for small bidders.

Example 2. *For the large bidder assume $\mu_L < 1$, and $t \sim U[0,1]$. For the small bidders assume $\mu_S = 1$ and $v \sim U[0,1]$. To construct an equilibrium, let $\widetilde{F}_S(v) = v/\mu_L$ for $v \in [0, \mu_L]$. The first-price auction with distributions F_L and $\widetilde{F}_S$ has equilibrium bid functions given by the expressions.*[11]

$$\begin{aligned}
B_D(t) &= \frac{\sqrt{1+kt^2}-1}{kt} \\
b_D(v) &= \frac{1-\sqrt{1-kv^2}}{kv} \\
k &= \mu_L^{-2} - 1 \\
\bar{b} &= \frac{\mu_L}{1+\mu_L},
\end{aligned}$$

where $\bar{b}$ is the maximum bid. For $v > \mu_L$ set $b_D(v) = \bar{b}$ to complete the description of equilibrium.

The large and small bidders' equilibrium payoffs are

$$\begin{aligned}
\Pi_D(t;\mu_L) &= \frac{t}{\sqrt{1+kt^2}}(t - B_D(t)) \\
\pi_D(v;\mu_L) &= \begin{cases} \frac{v}{\sqrt{1-kv^2}}(v - b_D(v)) & \text{if } v \leq \mu_L \\ v - \frac{\mu_L}{1+\mu_L} & \text{if } v > \mu_L. \end{cases}
\end{aligned}$$

In contrast, the bidders' payoffs in the UP auction are given by

$$\begin{aligned}
\Pi_U(t;\mu_L) &= \begin{cases} \frac{1}{2}t^2 & \text{if } t \leq 2\mu_L \\ \mu_L t - \mu_L^2 & \text{if } t > 2\mu_L \end{cases} \\
\pi_U(v;\mu_L) &= \begin{cases} v^2 & \text{if } v \leq \mu_L \\ v - \mu_L + \mu_L^2 & \text{if } v > \mu_L. \end{cases}
\end{aligned}$$

Note that both small and large bidders have reverse preferences over the UP and DP auctions.

Although we are able to construct examples where a small bidder prefers the DP auction to the UP auction (see Example **??** in the appendix), we also find that this is not typically the case and

[11] This example is from **?** who allows for the distributions to take the form $F(x) = x^a$ for $a > 0$, but it is sufficient for our purposes to work with the $a = 1$ case.

we provide a sufficient condition under which all small bidders prefer the UP auction to the DP auction. To introduce the sufficient condition, for the DP auction let $\phi_L(b)$ be large bidder's inverse bid function and $b(v)$ be the bid small bidders' equilibrium bid function. Then $\phi_L(b(v))$ is the type of large bidder that a type v small bidder ties with in the DP auction. Similarly, we $\tau(v)$ be the large bidder's inverse bid function in the UP auction. Thus, this is the type of the small bidder that the large bidder ties with if she bids v in the UP auction. The small bidder is weakly better off in the UP auction if

$$\int_{v_l}^{v} F_L(\tau(x))\, dx \geq \int_{v_l}^{v} F_L(\phi_L(b(x)))\, dx.$$

If this condition holds for all $v \in [v_l, v_h]$, then this is equivalent to saying that the distribution $F_L(\phi_L(b(x)))$ second-order stochastically dominates the distribution $F_L(\tau(x))$. We show that second-order stochastic dominance holds in this environment under the following sufficient condition which is easily verified for Example **??**.

Condition 1. *For $v \in [v_l, v_h]$, either*

(i) $\frac{F_L(\tau(v))}{\widetilde{F}_S(v)}$ is nonincreasing and $\frac{F_L(v)}{\widetilde{F}_S(v)}$ is nondecreasing; or

(ii) $\frac{F_L(v)}{\widetilde{F}_S(v)}$ is nonincreasing.

From Corollary **??** the small bidders always prefer the DP auction over the UP auction if they are the stronger bidder in the analogous first price auction, which explains the second part of the condition. This condition can be understood as a restriction on the degree of difference in strength. The argument for the small bidders' preference proceeds in two steps. First, we argue that under the condition $\tau(v)$ may only cross $\phi_L(b(v))$ once from above, and second that the mean of the random variable with distribution $F_L(\phi_L(b(x)))$ (or equivalently the highest bid made in the DP auction) is weakly larger than the mean of the random variable with distribution $F_L(\tau(x))$ (or the expected payment of the highest small type in the UP auction).

Proposition 2. *Under Condition **??**, the small bidders prefer the UP to the DP auction.*

Proof. If the second part of Condition **??** holds, the result follows from Corollary **??**. Therefore, assume the first part of the condition holds. We first show that if $\tau(v)$ and $\phi_L(b(v))$ cross, the latter is steeper. For a contradiction suppose that $\tau(v)$ and $\phi_L(b(v))$ cross at $\hat{v}$ and the former is steeper than the latter. Then at $\hat{b} = b(\hat{v})$, $\phi_L(\hat{b}) = \tau(\phi_S(\hat{b}))$ and $\phi_L'(\hat{b}) < \tau'(\phi_S(\hat{b}))\phi_S'(\hat{b})$. From the FOCs of the DP auction and Condition **??**,

$$\frac{F_L(\phi_L(\hat{b}))}{f_L(\phi_L(\hat{b}))}\frac{1}{\phi_S(\hat{b}) - \hat{b}} < \frac{\widetilde{F}_S(\phi_S(\hat{b}))}{\widetilde{f}_S(\phi_S(\hat{b}))}\frac{1}{\phi_L(\hat{b}) - \hat{b}}\tau'(\phi_S(\hat{b})) < \frac{F_L(\phi_L(\hat{b}))}{f_L(\phi_L(\hat{b}))}\frac{1}{\phi_L(\hat{b}) - \hat{b}},$$

which is a contradiction because the small bidders being weaker implies $\phi_S(b) < \phi_L(b)$ for all equilibrium bids (**?**, Proposition 3.5).

Next we show that the highest bid made in the DP auction is at least as big as the expected payment of v_h in the UP auction or that

$$\overline{b} = \int_{v_l}^{v_h} x \; dF_L(\phi_L(b(x))) \geq \int_{v_l}^{v_h} x \; dF_L(\tau(x)),$$

where $\overline{b}$ is the largest equilibrium bid in the DP auction. From the large bidder's preference for the Vickrey auction (due to her strength) and Condition **??**,

$$\overline{b} = \int_{0}^{\overline{t}} x \; d\widetilde{F}_S(\phi_S(B(x))) > \int_{v_l}^{v_h} x \; d\widetilde{F}_S(x) \geq \int_{v_l}^{v_h} x \; dF_L(\tau(x)).$$

Combined, the facts that $F_L(\tau(v))$ may only cross $F_L(\phi_L(b(v)))$ from below and that the payoff of the highest type of small bidder is higher in the UP auction prove the proposition. □

3.2 Revenue and Efficiency Ambiguity of UP and DP auctions

Ausubel et al. [2014] uses examples to show that the revenue and efficiency rankings of the UP and DP auctions are ambiguous. We obtain similar results in our large auction setting.

In Example **??**, we see that the DP auction gives greater revenue than the UP auction. This is immediate to see as the clearing price is the same in both auctions (the large bidder bids the same flat bid in both). Thus, the DP auction give higher revenue than the UP, as all small bidders who win units pay above the clearing price in the former, but not in the latter. In addition, when we set $\alpha = \frac{1}{2}$ in Example **??**, then the DP auction is efficient. This is because equilibrium bid behavior is equivalent to bid behavior in a symmetric first price auction. At the same time, the UP auction is inefficient because the large bidder engages in bid shading.

In Example **??** below, we illustrate that the two auctions have ambiguous revenue and efficiency rankings by constructing a case where the UP auction is more efficient (yields greater expected surplus) and has greater expected revenue.[12]

Example 3. *Suppose that $\mu_s = \mu_L = 1$ and the large bidder's value is t where $t \in \{0, 2\}$ each with probability $\frac{1}{2}$.[13] Small bidders values are uniformly distributed over $[0, 1]$. In the UP auction, the large bidder bids 0 if $t = 0$, and any bid greater than or equal to 1 is a best response if $t = 2$. Therefore, expected revenue is $\frac{1}{2}$. In the DP auction, the large bidder bids 0 if $t = 0$. Thus, all small bidders know that they win with probability of at least $\frac{1}{2}$ if they bid any amount $\epsilon > 0$. Moreover, no small bidder submits a bid above $\frac{1}{2}$ because*

$$\lim_{\epsilon \to^+ 0} \frac{1}{2}(v - \epsilon) = \frac{v}{2} \geq v - \frac{1}{2} \geq p(v - \frac{1}{2}) \;\forall p \in [0, 1], \; v \in [0, 1].$$

[12]Example **??** already illustrates the ambiguous efficiency rankings of the two auctions. While the DP auction generates greater ex-post surplus when $\alpha = \frac{1}{2}$, the UP auction generates greater ex-post surplus when $\alpha < \frac{1}{3}$.

[13]This violates our assumption that the large bidder has values that are draws of a random variable that has density f_L with full support over an interval. However, we could construct an almost equivalent example where the large bidder has type t that is the draw of a random variable that has an associated density f_L that has full support over $[0, 2]$, yet arbitrarily large density near 0 and 2 and arbitrarily small density over the interval $(\epsilon, 2 - \epsilon)$.

The large bidder never submits a bid above $\frac{1}{2}$ because small bidders never bid above $\frac{1}{2}$. Since small bidders never bid above their value, then an upper bound on the small bidders' bid function is $\overline{b}(v) = \min\{v, \frac{1}{2}\}$. Thus, if the large bidder has value $t = 0$, the upper bound on revenue is

$$\int_0^1 \overline{b}(v)dv = \frac{3}{8}.$$

If the large bidder has value $t = 2$, the upper bounds on revenue is $\frac{1}{2}$. Therefore, expected revenue of the DP auction bounded above by $\frac{7}{16}$.

Thus, here we have a case where the UP auction has higher revenue than the DP auction. Notice also that the outcome of the UP auction is Pareto efficient for any realization of the large bidder's value. If $t = 0$, the large bidder's value for additional units is below the value of all small bidders, and the large bidder does not win any units. If $t = 2$, the large bidder's value for additional units exceeds any small bidder's value for units, and the large bidder wins all units. At the same time the DP auction is not Pareto efficient, because when the large bidder has value $t = 2$ she does not win all units with probability one. Thus, there is a positive probability there are Pareto improving trades where the large bidder buys units from small bidder. This differs from the revenue and efficiency rankings presented in Example **??**, and it illustrates that the two auctions have ambiguous revenue and efficiency rankings.

3.3 Investment and Entry Incentives

In this section we study how the choice of auction formats influences the choices that bidders make prior to taking part in the auction. In particular, we study the large bidder's incentive to undertake a capacity-expanding investment, and small bidders' decision on entry.

3.3.1 Investment in Increased Capacity

We compare a large bidder's incentives for making costly capacity-expanding investments in the DP and UP auctions. Our analysis has similarities to the model in **?** of investment in single-unit auctions. In their model, one bidder may make an ex-ante investment that increases their value of winning.[14] **?** show that first-price auctions provide poor investment incentives relative to second-price auctions. When a bidder makes a value-enhancing investment in the first-price auction, it results in a negative indirect effect for the investor. After the investment, the investor's rivals know that the investor has a relatively stronger value distribution, and consequently the investor's rivals bid more aggressively.

The results of **?** and Lemma **??** also combine to imply that the DP auction provides the large bidder with poor incentives for making value-enhancing investments relative to a Vickrey auction. While considering firms' incentives for making value-enhancing investments is a natural way to

[14] To be precise, they assume the bidder is able to make a cost-reducing investment before participating in a procurement auction.

study investment in the single-unit context, there are other modes of investment that are important to study in the multi-unit setting, which have no analogue in a single-unit setting.

We study the large bidder's incentive for making a capacity-expanding investment. We show that the DP auction may provide more favorable incentives for making this type of investment than the UP auction does. Again, if the large bidder decides to invest, there is an indirect effect on small bidders' bids. Yet, the indirect effect of a capacity-enhancing investment in the DP auction is the opposite of the indirect effect identified by **?**. While they show that value-enhancing investments induce the investor's rivals to bid more aggressively, we show that capacity-expanding investments induce the investor's rivals to bid less aggressively.

More formally, we consider a model where the large bidder has the opportunity to make an investment that increases μ_L from some initial value. The large bidder makes her investment decision prior to realizing her value, and she knows the measure of small bidders she competes against (μ_S) prior to making the investment decision. After the investment decision is made, the small bidders observe the resulting μ_L and the auction proceeds as it does above. We assume that there is some cost associated with the investment, but for our analysis, we are only concerned with the marginal benefit of an increase in μ_L on the large bidder's ex-ante payoff and do not model a cost function.

Before considering the investment decision, observe that in a UP auction, when the large bidder has a relatively large initial capacity μ_L, then she may never exhaust her capacity in the auction (i.e. she will never buy μ_L units). Thus, any increase in the large bidder's capacity will not change the large bidder's bid behavior. Since a small bidder does not change her bid in the UP auction when the large bidder's capacity changes, we then see that an increase in μ_L would lead to no change in the large bidder's payoff in the UP auction. Or in other words, the demand reduction incentive in the UP auction is such that a large bidder may never use up her entire capacity, and thus, will have no incentive for making costly, capacity-expanding investments. We observe that this is the case in Example **??** when $\mu_L > 1/2$.

In the DP auction, thinking in terms of the corresponding $(F_L, \widetilde{F}_S)$ single-unit auction, an increase in the large bidder's capacity μ_L strengthens the distribution of small bidders in the competitive region ($\widetilde{F}_S$) by "stretching" it out over a bigger interval (i.e., v_h increases). When the large bidder has a higher capacity, the competitive region expands. If the large bidder previously had a capacity of μ_L units, then all small bidders with values that rank amongst the top $1-\mu_L$ measure of values would be considered sure winners. Yet, if the large bidder expands her capacity, the measure of bidders that are sure winners shrinks. Some small bidders who previously were sure winners now have values that are in the competitive region. Thus, the competitive region appears relatively stronger from the perspective of the large bidder, because she competes against small bidders with relatively higher values when she expands her capacity.

Proposition 1 of **?** shows that a strengthening of either distribution in this way causes both equilibrium bid distributions to become stronger in the corresponding $(F_L, \widetilde{F}_S)$ first-price auction.[15]

[15] Specifically, the proposition implies that an increase in μ_L, which strengthens $\widetilde{F}_S$, strengthens the equilibrium

In our model this is not surprising, because when the large bidders capacity expands competitive region expands to include small bidders with relatively higher values that previously were not in the competitive region because they were sure winners. By adding these higher value bidders into the competitive region, it is natural to expect that average bid in the competitive region will be greater.

At first glance, this effect may appear to have a negative impact on the large bidder's payoff. If it were true that the large bidder's payoff in the DP auction is identical to the "L" bidder's payoff in the corresponding $(F_L, \widetilde{F}_S)$ first-price auction, then Proposition 1 of **?** implies that the large bidder's payoff would decline after an increase in μ_L.

However, the large bidder's DP payoff is equal to the payoff from the $(F_L, \widetilde{F}_S)$ first-price auction multiplied by the capacity, and in our model capacity is increasing when the large bidder invests. Thus, we get distinct predictions. The large bidder's total payoff when she has capacity μ_L is

$$\Pi_D(B,t) = \mu_L \widetilde{F}_S(\phi_S(B))(t-B) = (1-\mu_S(1-F_S(\phi_S(B))))(t-B),$$

where we use ϕ_S for the small bidders' inverse bid function. The total effect of a change in μ_L can therefore be identified with its effect on the small bidders' inverse bid function ϕ_S alone, and this effect is not pinned down by the results of **?**.[16]

Our next lemma shows that in our case there is a consistent positive effect on ϕ_S after an increase in μ_L. In other words, given a small bidder's value, she places a lower bid when the large bidder's capacity increases. Thus, the average bid in the competitive region is increasing despite the small bidders bidding less. The smaller bids placed by the small bidders are offset by the stronger distribution of small bidders' values in the competitive region.[17]

Lemma 3. *Fix μ_S and let $0 < \mu_L < \hat{\mu}_L < 1$ be two distinct capacities. Let (ϕ_L, ϕ_S) and $(\widehat{\phi}_L, \widehat{\phi}_S)$ be respectively the equilibrium inverse bid functions of the $(F_L, \widetilde{F}_S)$ and $(F_L, \widehat{F}_S)$ first-price auctions, where $\widehat{F}_S$ is the analogue of $\widetilde{F}_S$ for $\hat{\mu}_L$. For any bid in the domain of both ϕ_i and $\widehat{\phi}_i$ with $i = L, S$, $\phi_L(b) > \widehat{\phi}_L(b)$ and $\phi_S(b) < \widehat{\phi}_S(b)$.*

Proof. Let $[b_l, b_h]$ and $[b_l, \hat{b}_h]$ be the domains of (ϕ_L, ϕ_S) and $(\widehat{\phi}_L, \widehat{\phi}_S)$ respectively. Lemma 5 of **?** implies that $\hat{b}_h > b_h$, while their Proposition 1 implies that $\phi_L(b) > \widehat{\phi}_L(b)$ and $\widetilde{F}_S(\phi_S(b)) > \widehat{F}_S(\widehat{\phi}_S(b))$ for all $b \in (b_l, b_h) \equiv \mathcal{I}$. Using the FOC of bidder L, we find that $F_S(\widehat{\phi}_S(b))/F_S(\phi_S(b))$ is

bid distributions of both bidders in the first-price auction. That is, with ϕ_i representing the inverse bid functions, $F_L(\phi_L(b))$ and $\widetilde{F}_S(\phi_S(b))$ decrease pointwise on the interior of their common support. Both bidders in the first-price auction therefore have to pay more for the same probability of winning and are thus worse off.

[16] From the result that the bid distributions become stronger we can conclude that ϕ_L falls for any bids that are common to the two equilibria because F_L is constant, but the **?** result does not pin down changes in ϕ_S since $\widetilde{F}_S$ is not constant.

[17] There is no contradiction here with **?**. Their results imply that $\widetilde{F}_S(\phi_S(b))$ becomes stronger when $\widetilde{F}_S$ does. Ours show that in this case while $\widetilde{F}_S$ is becoming stronger the small bidders are shading more (i.e., ϕ_S increases).

increasing on $\mathcal{I}$, because given some $b^* \in \mathcal{I}$

$$\frac{F_S(\widehat{\phi}_S(b))}{F_S(\phi_S(b))} = \frac{F_S(\widehat{\phi}_S(b^*))}{F_S(\phi_S(b^*))} \exp\left\{\int_b^{b^*} \frac{1}{\phi_L(x)-x} - \frac{1}{\widehat{\phi}_L(x)-x}\,dx\right\},^{18}$$

and the expression in brackets is strictly increasing in b. If $\phi_S(b) = \widehat{\phi}_S(b)$ for some $b \in \mathcal{I}$ then $\phi'_S(b) < \widehat{\phi}'_S(b)$, so $\phi_S(b^*) > \widehat{\phi}_S(b^*)$ implies that $\phi_S(b) > \widehat{\phi}_S(b)$ for all $b \in (b_l, b^*)$. That is $\widehat{\phi}_S(b)$ crosses $\phi_S(b)$ at most once and from above.

To generate a contradiction suppose that there exists a b^* such that $\phi_S(b) > \widehat{\phi}_S(b)$ for all $b \in (b_l, b^*)$. First we have that $F_S(\widehat{\phi}_S(b_l))/F_S(\phi_S(b_l)) = 1$, which follows from the argument given in the proof of Theorem 3 in **?**.[19] Then for all b close to b_l, $F_S(\widehat{\phi}_S(b))/F_S(\phi_S(b)) < F_S(\widehat{\phi}_S(b^*))/F_S(\phi_S(b^*)) < F_S(\widehat{\phi}_S(b_l))/F_S(\phi_S(b_l))$, which is impossible given the continuity of both bid distributions. □

Remark: This lemma is a statement about the effect of lengthening the support of one bidder in an asymmetric first-price auction with two bidders and is thus applicable to single-unit auctions as well.

We use Lemma **??** to show that the large bidder is always made better off from capacity expansion in the DP auction. Despite the fact that the small bidders' bid distribution in the $(F_L, \widetilde{F}_S)$ first-price auction becomes stronger, the preceding argument and the lemma show that the net effect is to make the large bidder better off, because the small bidders bid less aggressively.

Proposition 3. *Under the conditions of Lemma* **??***, in the DP auction the large bidder is strictly better off after an increase in her capacity.*

We use Proposition **??** to argue that the large bidder has greater marginal incentives to increase capacity in the DP auction versus the UP auction when her initial capacity μ_L is sufficiently large. Proposition **??** tells us that the large bidder always benefits from increased capacity in the DP auction. However, the large bidder only has an incentive to grow her capacity in the UP auction if her initial capacity μ_L is sufficiently small. More formally, note that if a large bidder uses all of her capacity in the UP auction, she gets an interim payoff of

$$\Pi_U(t) = \mu_L(t - F_S^{-1}(\mu_L)).$$

In other words, she wins μ_L units and the clearing price is the highest value associated with a losing small bidder $F_S^{-1}(\mu_L)$. Thus, the large bidder's marginal benefit associated with increased capacity in the UP auction in equilibrium is

[18] This expression comes from integrating bidder L's FOC backwards from b^* in each auction. Note that there is a μ_L and a $\hat{\mu}_L$ that cancel.

[19] That $F_S(\widehat{\phi}_S(b_l))/F_S(\phi_S(b_l)) = 1$ is immediate if at the lowest type that submits a bid $F_S > 0$. Otherwise, we appeal to the argument in Lemma 5 in **?**, who analyze an analogous problem and show that $\lim_{b \downarrow b_l} \widehat{\phi}'_S(b) = \lim_{b \downarrow b_l} \phi'_S(b)$.

$$\frac{\partial \Pi_U(t)}{\partial \mu_L} = \max\{(t - F_S^{-1}(\mu_L)) - \mu_L \frac{d}{d\mu_L} F_S^{-1}(\mu_L), 0\}.$$

If the large bidder's type is such that she gets no benefit from utilizing the increased capacity, then there is no additional benefit from purchasing capacity. Hence, we add the max when defining marginal benefit. We show that when the large bidders capacity is sufficiently large (i.e. sufficiently close to 1), then the marginal benefit of increased capacity in the UP auction is zero.

To see this, note that $F_S^{-1}(q)$ is continuous and strictly increasing in q because we assume F_S is continuous and f_S has full support. Thus, $\lim_{\mu_L \uparrow 1} F_S^{-1}(\mu_L) = 1$, because 1 is the upper bound on the small bidder's support. Or equivalently,

$$\lim_{\mu_L \uparrow 1} t - F_S^{-1}(\mu_L) \leq 0, \forall t \in [0, 1].$$

In addition, $-\mu_L \frac{d}{d\mu_L} F_S^{-1}(\mu_L) = -\mu_L \frac{1}{f(F_S^{-1}(\mu_L))} < 0,\ \forall \mu_L > 0$. Thus, there exists a $\mu_L^* < 1$ such that for $\mu_L > \mu_L^*$,

$$(t - F_S^{-1}(\mu_L)) - \mu_L \frac{d}{d\mu_L} F_S^{-1}(\mu_L) < 0,\ \forall t \in [0, 1].$$

This implies that the marginal benefit of buying additional capacity is zero for all types of large bidder when her initial capacity exceeds $\mu_L^* \in (0, 1)$. When combined with Proposition **??**, this observation yields Proposition **??**.

Proposition 4. *If μ_L is large enough to make all types of large bidder unconstrained in the UP auction, the ex-ante marginal incentives to invest in capacity (increase μ_L) are stronger in the DP auction.*

Note that "large enough" does not imply that μ_L is large. For instance, in Example **??** all types of large bidders are unconstrained in the UP auction when $\mu_L > 1/2$, which is the quantity desired by the $t = 1$ large bidder.

3.3.2 Entry of Small Bidders

Next we extend the model to study ex-ante entry of small bidders. Thus, we allow μ_S to be endogenous. We suppose that there is an unlimited supply of small bidders that may, before learning their values but given μ_L, decide whether to participate in the auction after paying an upfront cost c. We assume that small bidders in this environment enter until they exhaust their ex-ante net profits from entering. The entry decision determines the measure of small bidders, μ_S, that participate in the auction. As in the previous extension, after small bidders make their entry decisions all bidders observe the measure of small bidders that enter μ_S and the auction proceeds as in the original model.

A consequence of the small bidders' preference for the UP auction is that in this entry model the UP auction induces more small bidders to enter than the DP auction does. To see this, suppose that μ_S^{DP} is the equilibrium level of entry in the DP auction. Thus, a small bidder's expected payoff from participating in the DP auction is c. Proposition **??** then implies that the expected payoff for a small bidder from participating in the UP auction exceeds c when the measure of small bidders is μ_S^{DP}. We are then able to establish that there is more entry in the UP auction by showing that the benefits of entry for small bidders decrease as more small bidders enter. That is, the small bidders expected payoff from participating in the auction falls when there is more competition from other small bidders.

Condition **??** is sufficient to show that a small bidder's payoff from participating in the auction falls as μ_S increases. In particular, the clearing price in the UP auction rises in response to an increase in the measure of small bidders, and thus the small bidders ex-ante payoff must be decreasing in this measure as well. To see that the clearing price rises as μ_S increases, note that the elasticity of the large bidder's residual supply curve in the UP auction is given by

$$\frac{S'(B)B}{S(B)} = \frac{\mu_S f_S(B)B}{1-\mu_S(1-F_S(B))}. \tag{1}$$

It is easy to verify that this is increasing in μ_S and that this implies that the bid chosen by the large bidder, and hence the clearing price, is increasing in μ_S for all types of large bidder.[20] Hence we have the following.

Proposition 5. *If Condition* **??** *holds there is more entry by small bidders in the UP auction.*

4 Aggregate Demand Uncertainty

Our prior results show that the large bidder gets a higher payoff in DP auction than the UP auction. In these results, we use the fact that the large bidder knows the distribution of small bidder types, implying that in equilibrium the large bidder is able to perfectly predict the residual supply curve. In this section we test the robustness of our predictions by allowing for aggregate demand uncertainty over the small bidders' value distribution. We show that aggregate uncertainty improves the large bidder's payoff in the UP auction but not necessarily in the DP auction. An example shows that the large bidder's auction preference changes from the DP auction to the UP auction when there is aggregate uncertainty and a large degree of asymmetry in the distributions of values. When the strength of the large and small bidders' distribution are equal, which one may think of as a symmetric case, aggregate uncertainty cannot lead to a preference reversal.

4.1 Model with Aggregate Uncertainty

We add aggregate uncertainty by assuming that the distribution of small bidder types is stochastic. We assume that small bidders have values that are in the interval $[0,1]$ and are distributed according to $F_S(v|s)$, where s is a parameter distributed according to the absolutely continuous distribution $G(s)$ with support $[\underline{s}, \overline{s}]$ and density $g(s)$.[21] We assume that s orders the distributions in terms of

[20] The large bidder sets the clearing price to be such that $\frac{S'(B)B}{S(B)} = \frac{B}{t-B}$.

[21] The joint density of v_S and s is therefore $f_S(v_S|s)g(s)$.

reversed hazard rate dominance, or that $s' > s$ implies $f_S(v|s)/F_S(v|s) \geq f_S(v|s')/F_S(v|s')$ for all v. This gives the parameter s the interpretation that it measures the "strength" of the small bidders with lower values indicating a stronger distribution. For tractability we assume that $\mu_L = \mu_S = 1$ in this section. Example **??** illustrates a distribution of small bidders that fits within this setting.

Example 4.

$$\begin{aligned} F_S(v|s) &= v^{1-s} \\ G(s) &= s \quad 0 \leq s \leq 1 \\ E_S[F_S(v|S)] &= \frac{1-v}{-\log(v)} \end{aligned}$$

In what follows we compare the large bidders preference toward the DP and UP auctions. In order to characterize bid behavior in the DP auction, we consider the expected distribution of small bidder's values $E_S[F_S(\cdot|S)]$. In addition, we show how increases in the amount of aggregate demand uncertainty influence the large bidder's preference for auction formations. To formalize this, given two distributions of s, $G_1(s)$ and $G_2(s)$, call G_1 a *mean-preserving spread* of G_2 if the expected distribution of values is the same for every v under G_1 and G_2 but the variance, $E_S[F_S(\cdot|S)^2] - E_S[F_S(\cdot|S)]^2$, is weakly greater for G_1 at all v. In Example **??**, $G_1(s)$ is a mean preserving spread of $G_2(s)$.

Example 5. $\widehat{F}_1(v) \geq \widehat{F}_2(v)$ *are two distribution functions on* $[0, 1]$.

$$\begin{aligned} F_S(v|s) &= s\widehat{F}_1(v) + (1-s)\widehat{F}_2(v) \\ G_1(s) &= s \quad 0 \leq s \leq 1 \\ G_2(s) &= 2s - \frac{1}{2} \quad \frac{1}{4} \leq s \leq \frac{3}{4} \\ E_S[F_S(v|S)] &= \frac{1}{2}\left(\widehat{F}_1(v) + \widehat{F}_2(v)\right). \end{aligned}$$

4.2 Uniform-Price Auction

In the UP auction, the small bidders optimally bid up to their value, so the amount won by the large bidder for a given bid B and realization s is $q(B, s) = F_S(B|s)$, which can be interpreted as a residual supply curve. We describe the large bidder's strategy as mapping each s to a bid $B(t, s)$. Given the ordering of s if $B(t, \cdot)$ is nonincreasing, the mapping from quantities to bids is downward sloping. Since the auction rules require the bids to be a function of quantities, we also require that $F_S(B(t, s)|s)$ is a nondecreasing function of s, because otherwise we might have two distinct bids assigned to the same quantity.

The large bidder's objective is

$$\max_B \int_{\underline{s}}^{\overline{s}} F_S(B(t,s)|s)(t - B(t,s))dG(s)$$
$$\text{s.t. } B(t,\cdot) \text{ nonincreasing}$$
$$F(B(t,\cdot)|\cdot) \text{ nondecreasing.}$$

If we temporarily ignore the constraints, the bid function $B(t,s)$ that maximizes the large bidder's objective pointwise (i.e., for each s) satisfies

$$\frac{f_S(B|s)}{F_S(B|s)}(t - B) = 1. \tag{2}$$

Figure **??** looks at the setting from Example **??**. The figure illustrates the large bidder's best response as described by Equation (**??**) for varying aggregate states. The left-hand side of Equation (**??**) is decreasing in s given our assumptions, and this ensures that the B that solves (**??**) is a decreasing function of s for a fixed t, meaning that the first constraint is slack.

The solution for Example **??** is

$$B(t,s) = \frac{1-s}{2-s}t$$
$$q(s) = \left(\frac{1-s}{2-s}t\right)^{1-s}.$$

It is straightforward to check that $q(s)$ is increasing in this solution, satisfying the second constraint.

If for a proposed solution $B(t,s)$ $F_S(B(t,\cdot)|\cdot)$ is decreasing over some interval, $[s_1, s_2]$, then the solution to the optimization problem involves ironing over a containing interval, on which $F_S(B(t,\cdot)|\cdot)$ would be constant. This would result in a downward corresponding jump in the bid function. Even in this case the large bidder chooses a different clearing price in each state with higher states corresponding to lower clearing prices.

Proposition 6. *The large bidder's equilibrium bid in the UP auction with aggregate uncertainty results in a clearing price that is decreasing in the state variable s. If $F_S(B(t,\cdot)|\cdot)$ is a weakly increasing function when $B(t,s)$ solves (**??**), then the large bidder maximizes her objective pointwise for each (t,s).*

It is feasible for the large bidder to submit a flat bid curve that assigns the same bid to each state t. In this case, the large bidder's payoff would be

$$\int_{\underline{s}}^{\overline{s}} F_S(B|s)\,dG(s)(t - B) = E_S[F_S(B|S)](t - B).$$

In other words, the large bidder could bid as if she faced the average supply curve $E_S[F_S(B|S)]$, but she does not because her optimal strategy gives a higher payoff than submitting a flat bid. That is,

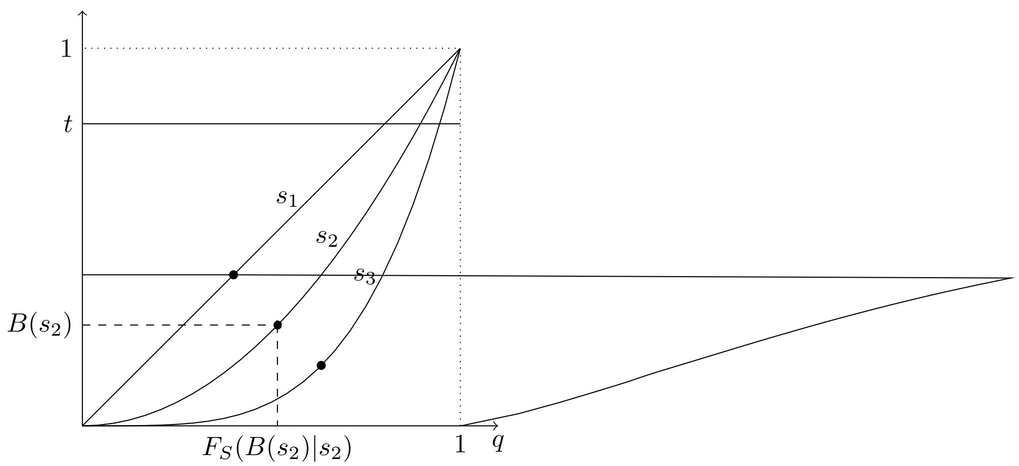

Figure 2: Bid Function in Example **??**

if we add uncertainty to the residual supply curve, the large bidder can profit from this uncertainty in the UP auction. This occurs because the UP auction allows the large bidder to specify a clearing price for each s that does not affect the clearing prices for other realizations of s.

This argument can be extended beyond comparisons between a distribution $G(s)$ and its mean, $E_S[F_S(B|S)]$. The key observation is that the large bidder's equilibrium ex-post payoff, $F_S(B^*(t,s)|s)(t-B^*(t,s))$, is convex in the state s, due to the envelope theorem and $F_S(v|s)$ increasing in s. The large bidder's payoff in the UP auction can therefore be seen as the expected value of a convex function of s and is thus made higher under the distribution $G_2(s)$ compared to $G_1(s)$ when $G_2(s)$ is a mean-preserving spread of $G_1(s)$.

Proposition 7. *If $G_2(s)$ is a mean-preserving spread of $G_1(s)$, then the large bidder's equilibrium payoff in the UP auction is higher when S is distributed according to $G_2(s)$.*

This proposition formalizes the idea that the large bidder benefits from aggregate uncertainty in the UP auction.

4.3 Discriminatory-Price Auction

In contrast, the DP auction rules do not typically allow the large bidder to profit from uncertainty. The DP auction requires the large bidder to attach a payment to each *quantity* increment, applying in all states of the world in which she wins that increment. This rule leads to the general phenomena of "bid flattening" (**?**). In this section we show that the large bidder's bid may be constant and hence completely insensitive to the aggregate uncertainty.

Analysis of the DP auction is potentially complicated for a couple of reasons. The first is that the small bidders' equilibrium strategies are endogenous and depend on the large bidder's strategy. Second, in principle, the small bidders' inferences about the state of the aggregate uncertainty based on their value may be important. However, the incentives in the DP auction favor flatter bid curves for the large bidder to the extent that the large bidder optimally submits a flat demand curve under certain conditions. Informally, this effect can be seen clearly in a special case in which the residual supply curve is a step function with N evenly divided steps,[22] each of which is the realization of an order statistic from N independent draws from a distribution $H(\cdot)$. In this case the large bidder's payoff when she submits a bid schedule $B(t,q)$ is

$$\sum_{i=1}^{N} \frac{1}{N} H_{(N-i+1)}(B(t,i/N))(t - B(t,i/N)),$$

where $H_{(N-i+1)}$ is the distribution of the $(N-i+1)^{th}$ order statistic. She wins the $(N-i+1)^{th}$ increment if and only if $B(t,i/N)$ is greater than the $(N-i+1)^{th}$ order statistic. The fact that $h_{(N-i+1)}(x)/H_{(N-i+1)}(x)$ is increasing in i means that the monotonicity constraint on her bid binds for every quantity and her optimal constrained bid is flat, which incidentally means that her payoff becomes $H(B(t))(t - B(t))$, or the payoff from a first-price auction.

From a small bidder's perspective, if the large bidder submits a flat bid, then the large's bid will always be the lowest winning bid ex post, and in this environment predicting this clearing price will be the only inference that that small bidders will need to make. In other words, a small bidder's payoff is a function of the large bidder's type distribution and bid function alone.

To formulate the large bidder's payoff in the DP auction, consider her bids as a function of her value and a quantity, $B(t,q)$. Suppose that the small bidders bid according to an increasing $b(v)$ with inverse $\phi_S(b)$. For each B and q we can find a $s(B,q)$,

$$s(B,q) = \begin{cases} \inf\{s | q \leq F(\phi_S(B)|s)\} & q \leq F(\phi_S(B)|\overline{s}) \\ \overline{s} & q > F(\phi_S(B)|\overline{s}), \end{cases}$$

so that whenever $S > s(B(t,q),q)$, the large bidder wins the q^{th} increment for a price $B(t,q)dq$. We can then write her expected payoff as

$$\int_0^1 (1 - G(s(B(t,q),q)))(t - B(t,q))\, dq. \tag{3}$$

Note that there are increments that the large bidder always wins and never wins (i.e., there are intervals of quantities over which $G(s(B(t,q),q)) \in \{0,1\}$).

Using the analogy to the order statistic example above, we give a sufficient condition under which the large bidder with type t optimally submits a flat bid schedule, which is satisfied by the examples given above if for example $\widehat{F}_1(v) = v$ and $\widehat{F}_2(v) = v^2$.

Condition 2. *For all q, $B \in [0,t]$, and increasing $\phi_S(B)$,* $\frac{\partial^2}{\partial B \partial q} \ln(1 - G(s(B,q))) \geq 0$.

To evaluate the partial derivatives of $s(B,q)$, one can implicitly differentiate $q = F(\phi_S(B)|s)$, noting that $\frac{\partial}{\partial s} F(\phi_S(B)|s) > 0$ by assumption. Condition **??** then places restrictions on the relationship between F and G.

[22]This residual supply curve should take into account the bid function used by the small bidders. This informal argument is meant to convey the intuition motivating the subsequent analysis.

If we think of $H(B|q) = (1 - G(s(B,q)))$ as the probability that a large bidder's bid of B on the q^{th} increment wins, then this condition says that $h(B|q)/H(B|q)$ is nondecreasing in q, or that the bid distribution for subsequent increments becomes stochastically stronger in the reverse hazard rate sense (as is the case in the order statistic example above). Condition **??** can be simplified if we assume $G(s)$ has an increasing hazard rate.[23] Since $s(B,q)$ is increasing in q, Condition **??** is satisfied if the following holds.

Condition 3. *$G(s)$ has an increasing hazard rate and $s_{Bq}(B,q) \leq 0$.*

Another way to understand this condition is that it ensures that a single-crossing condition is satisfied for the large bidder's payoff on each increment.[24]

Lemma 4. *If Condition* **??** *holds, the large bidder's best response to some increasing bid function $b(v)$ is to use a flat bid schedule.*

Proof. Consider choosing B to maximize (**??**) pointwise. Let $\pi(t,B,q)$ stand for the payoff on the q^{th} increment. Condition **??** ensures that $\pi(t,B,q)$ has increasing differences in (B,q) and hence that for any t a selection $B(q) \in \arg\max \pi(t,B,q)$ is nondecreasing in q (**?**). When this occurs, the only possible solution to the large bidder's problem with the monotonicity constraint on bids is to submit a flat bid schedule. □

Submitting a flat bid schedule makes the large bidder's problem equivalent to one without aggregate demand uncertainty, where she faces the expected residual supply curve, and decides on a price-quantity pair. Her objective after this simplification can be written as

$$\max_B \mathrm{E}_S[F_S(\phi(B)|S)](t - B), \tag{4}$$

where B is now a scalar. The following result is immediate.

Proposition 8. *If Condition* **??** *holds, the large bidder's equilibrium payoff in the DP auction is unchanged from the case without aggregate uncertainty in which small bidder's values are distributed according to* $\mathrm{E}_S[F_S(v|S)]$.

Therefore, compared to the case with no aggregate uncertainty, aggregate uncertainty may improve the large bidder's payoff in the UP auction while leaving her payoff in the DP auction unchanged. Yet, Proposition **??** shows that without aggregate demand uncertainty, the large bidder prefers the DP auction over the UP auction. This leads to our next question: if the presence of aggregate demand uncertainty increases the large bidders payoff from a UP auction, but not a DP auction, then can aggregate demand uncertainty lead to a preference reversal for the large bidder? We show that a preference reversal can occur only if the large bidder is sufficiently strong. Thus,

[23] $\frac{g(\cdot)}{1-G(\cdot)}$ is an increasing function.

[24] Note that taking this perspective, we can relax the constant marginal values assumption for the large bidder without affecting the following results as long as $\frac{\partial^2}{\partial B \partial q} \ln\left[(1 - G(s(B,q)))(V(q,t) - B)\right] \geq 0$, where $V(q,t)$ is the marginal value of the q^{th} increment.

if the large bidder is (weakly) weaker than the small bidder in the sense of the reverse hazard rate, then the large bidder prefers the DP auction, even when there is aggregate demand uncertainty.

More formally, we compare the distribution of the large bidder's value with the ex-ante distribution of small bidder values $F_S(v)$ where $F_S(v) = \mathrm{E}_S[F_S(v|S)]$. Under Condition **??**, we see that the outcome of the equilibrium bid strategies of the large bidder and small bidders are the same as the bid strategies in the setting without aggregate demand uncertainty, where the small bidders have distribution $F_S(v)$. Thus, the expected quantity won by the large bidder when she is type v and participates in a DP auction with aggregate demand uncertainty against small bidders distributed according to $F_S(v|s)$ is the same the quantity the large bidder wins in a DP auction without aggregate demand uncertainty when she is type v and small bidders are distributed according to $F_S(v)$.

We then show that the large bidder prefers the DP auction to the UP auction by comparing the large bidder's preference for each to the Vickrey auction. The Vickrey auction yields an efficient allocation, even with aggregate demand uncertainty. In the UP auction, the large bidder decides the clearing price in each state of the world. The clearing price is also lower than the efficient price, due to demand reduction. Thus, the large bidder wins fewer units in the UP auction than in the Vickrey auction. If $q_L^V(v)$ is the expected number of units that the large bidders wins at the interim stage in the Vickrey auction and $q_L^{UP}(v)$ is the expected number of units in the UP auction, then $q_L^V(v) \geq q_L^{UP}(v)$. An envelope theorem argument then implies that the large bidder prefers the Vickrey auction to the UP auction with aggregate demand uncertainty.

We can similarly compare the large bidder's expected payoff in the DP and Vickrey auctions. Recall bid behavior in the DP auction is unchanged by the presence of aggregate demand uncertainty. Thus, using Lemma **??** we see equilibrium bid behavior can be characterized by studying an asymmetric first-price auction. If the large bidder's type distribution F_L is weakly weaker than the small bidders' type F_S, then **?** show that $B(v) \geq b(v)$, or that the large bidder is more aggressive than the small bidder. Thus, the large bidder defeats all small bidders who have lower values than her value. Or equivalently, the large bidder wins weakly more units than she would in the efficient outcome, $q_L^{DP}(v) \geq q_L^V(v)$, where q_L^{DP} is the expected number of units won in the DP auction.[25] Therefore, the envelope theorem implies the following.

Proposition 9. *If Condition* **??** *holds and F_L is weakly weaker than F_S, then the large bidder prefers the DP auction to the Vickrey auction, and the Vickrey auction to the UP auction.*

If the large bidder is sufficiently strong, and there is a sufficiently large amount of aggregate demand uncertainty, then large bidder's preference for the two auctions can reverse. As the large bidder becomes stronger, her interim expected per-unit payoff falls in the DP auction. This is shown by **?** in the context of first-price auctions, and Lebrun's result carries over to our setting using Lemma **??**. We use this to motivate the construction of Example **??**, where we show that it is possible for the large bidder's preference for the two auction to reverse in an extreme case.

[25] The reverse relationship would hold if the large bidder was stronger than the small bidder.

Example 6. *Small bidders' values are perfectly correlated and from the ex-ante perspective, uniformly distributed.*

$$F_S(v|s) = \begin{cases} 0 & if\ v < s \\ 1 & if\ v > s \end{cases},$$

$$G(s) = s.$$

The large bidder has value 1, which is commonly known.

In the UP auction, the large bidder always bids 1*, because ex post the supply curve is perfectly elastic. The market clearing price is equal to* s *and the large bidder wins all units for price* s*. Thus, in expectation the large bidder gets payoff of* 1/2.

In the DP auction, since her value is known, there is no equilibrium in which the large bidder plays a pure strategy. For a given mixed strategy let $\underline{B}$ *be the lowest bid that that the large bidder bids. Then* $b(\underline{B}) = \underline{B}$ *in equilibrium. Thus, the large bidder gets payoff*

$$F_S(\underline{B})(1 - \underline{B})$$

by bidding $\underline{B}$ *. We then have that*

$$\underline{B} = \arg\max F_S(b)(1 - b) = \frac{1}{2}.$$

This holds because we know the quantity the large bidder wins when bidding B*,* $q(B)$ *is such that* $q(B) \geq F(B)$ *because* $b(v) \leq v$*. Thus, if* $\underline{B} \neq \arg\max F_S(b)(1-b)$*, the large bidder could increase her payoff by bidding*

$$F(\underline{B})(1 - \underline{B}) < F(\frac{1}{2})(\underline{\theta} - \frac{1}{2}) \leq q(\frac{1}{2})(1 - \frac{1}{2}).$$

Thus, the large bidder randomizes, and the lowest bid she submits is $\underline{B} = \frac{1}{2}$*. When the large bidder bids* $\frac{1}{2}$*, the expected payoff is* $\frac{1}{4}$*. Thus, the large bidder has an expected payoff of* $\frac{1}{4}$ *in the DP auction.*

Notice also this example illustrates a case where small bidders prefer the DP auction. In the UP auction, the small bidders are guaranteed a payoff of zero. In the DP auction, small bidders win with positive probability when $s > \frac{1}{2}$.

5 Conclusion

We introduce a tractable model of large multi-unit auctions with a single large bidder. We show that in this situation the large bidder has a clear preference for a discriminatory pricing rule compared to a uniform pricing rule. This preference also encourages her to invest more in increases in capacity in the discriminatory-price auction as long as her capacity is not too small. We require additional conditions on the distribution of values to conclude that the small bidders have the reverse preference for the uniform-price auction. When this condition holds, we show that the uniform-price auction also incentivizes more entry than the discriminatory-price auction. As in the existing models of multi-unit auctions, we find that the revenue and efficiency comparison between the uniform-price and discriminatory-price auctions is generally ambiguous.

Our model is tractable in comparison to more general models of multi-unit auctions. This is because the analysis of the uniform-price auction is straightforward and we are able to reduce the discriminatory-price auction to a asymmetric first-price auctions for a single-unit. Therefore, this model may prove fruitful in future research.

A Proofs

Proof of Lemma ?? Let $\widetilde{F}_S(x) = \frac{1}{\mu_L} - \frac{\mu_S}{\mu_L}(1 - F_S(x)))$. This is a distribution function for an appropriately defined support, $[v_l, v_h]$. If $\mu_S > 1$ choose v_l to solve $\widetilde{F}_S(v_l) = 0$, setting $v_l = 0$ otherwise. Note $\widetilde{F}_S(x)$ has a mass point at $x = 0$ if $\mu_S < 1$. Choose v_h to solve $\widetilde{F}_S(v_h) = 1$.

Temporarily ignore the small bidders with valuations outside of the interval $[v_l, v_h]$. Observe that it is a weakly dominant strategy to submit a flat bid curve for the large bidder,[26] and consider the first-price single-unit auction with two bidders in which the type distributions are F_L and $\widetilde{F}_S$. An equilibrium of this auction exists in which there is a common maximum bid $\bar{b}$ (**???**). Let $(B, \widetilde{b})$ be the equilibrium bid functions. Define b on $[0, 1]$ by setting $b(v) = \widetilde{b}(v)$ for $v \in [v_l, v_h]$, $b(v) = v$ for any $v \leq v_l$ and $b(v) = \bar{b}$ for any $v \geq v_h$. We claim that this is an equilibrium of the original game. For small bidders with $v < v_l$, any bid above their value is weakly dominated. For bidders with $v \in [v_l, v_h]$ the conditions for $(B, \widetilde{b})$ to be an equilibrium in the first-price auction game ensure that no deviation in $[v_l, \bar{b}]$ is profitable. In particular, bids above $\bar{b}$ are weakly dominated by $\bar{b}$ which wins with probability one. Bids below v_l lose with probability one. For small bidders with $v > v_h$, the single crossing condition on their payoff ensures that they earn more from bidding $\bar{b}$ and winning for sure than by bidding at any lower level. Finally, the mass of small bidders that may arise at the upper end of the bid distribution does not cause any difficulties for the large bidder's proposed strategies because with a bid of $\bar{b}$ she wins all of the units that she has value for with probability one, and so cannot gain by increasing her bid.

Example 7. *Suppose that $\mu_L = \mu_S = 1$ and let $F_L(t)$ be such that there is a $\frac{1}{2}$ probability that the large bidder is type $t = 0$ and there is a $\frac{1}{2}$ probability that the large bidder is type $t = 10$. Small bidders have type $F_S(t) = (1 - \epsilon)t$ if $t < 1$. There is also an $\epsilon > 0$ measure of small bidders with type $t = 100$.*

In the UP auction, small bidders truthfully report their type and the large bidder best responds by bidding 0 if $t = 0$ and 1 if $t = 10$. Thus, a small bidder with type $t = 100$ gets an expected payoff of 99.5.

In the DP auction no bidder bids above their value. Thus, for a small bidder, and bid $b > 0$ wins with probability of at least $\frac{1}{2}$. This implies that any small bidder with type $v \leq 1$ bids $b(v) \leq \frac{1}{2}$

[26] Any decreasing bid curve that crosses the residual supply curve at the same point as a flat bid curve must lead to a lower payoff for the When the same fraction of units are won at a higher cost.

because

$$\lim_{b\to 0}\frac{1}{2}(v-b)\geq v-\frac{1}{2},$$

where the left hand side is a lower bound a bidder's payoff from submitting an arbitrarily small bid and the right hand side is an upper bound on a bidder's payoff from bidding $\frac{1}{2}$*.*

Moreover, when the large bidder has type $t = 10$*, she bids by submitting a flat bids that mixes over* $(0, \overline{b})$ *with no atoms, where* $\overline{b}$*. The large bidder mixes over a support that has zero as a lower bound, because if the large bidder mixed over a support where* $\underline{b} > 0$ *was the lower bound on her bid, then no small bidder would bid in the interval* $(\epsilon, \underline{b})$*. In addition,* $\overline{b} \leq \frac{1}{2}$ *because almost all small bidders bid below* $\frac{1}{2}$*.*

Since the large bidder mixes over $(0, \overline{b})$ *with no atoms, then it follows that small bidders bid according to strategy* $b(v)$*, that is continuous and such that* $b(0) = 0$ *and* $b(1) = \overline{b}$*. Thus, a small bidder's interim probability of winning given that she is type* v *is* $q(v)$ *where* q *is continuous, and such that* $q(v) \geq \frac{1}{2}$ *if* $v > 0$ *and* $q(1) \approx 1$*. A standard envelope theorem condition then implies that a small bidder with type* $v = 1$ *gets payoff*

$$1-\overline{b}\approx q(1)(1-\overline{b})=\int_0^1 q(s)ds>\frac{1}{2}\implies \overline{b}<\frac{1}{2}.$$

Thus, a small bidder with type $v = 100$ *knows that the clearing price is at most* $\overline{b}$*. Thus, the bidder with type* $v = 100$ *gets a payoff of at least* $100 - \overline{b} > 99.5$ *because* $\overline{b} < \frac{1}{2}$*.*