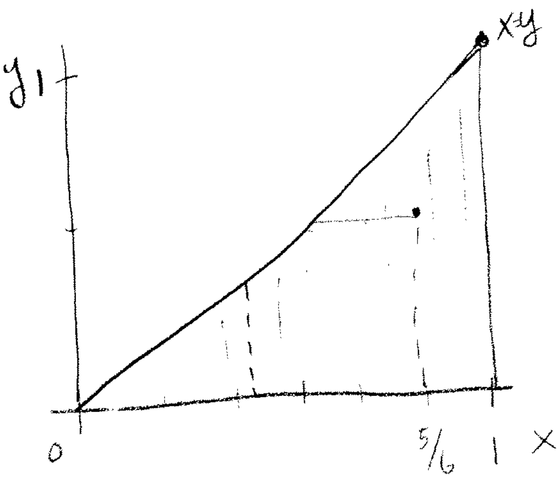
Practice with Jt. Prob Dists. - Solutions

1. a. To have a valid pdf, we need



$$\int_0^1 \int_0^x k(x-y)\,dy\,dx = 1$$

$$\Rightarrow k\int_0^1\int_0^x (x-y)\,dy\,dx = k\int_0^1 xy - \tfrac{1}{2}y^2\Big|_0^x dx$$

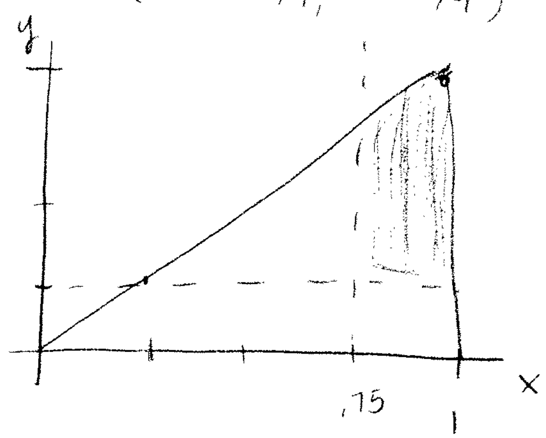
$$= k\int_0^1 \tfrac{1}{2}x^2\,dx = k\left(\tfrac{1}{6}x^3\Big|_0^1\right) = \frac{k}{6} = 1 \qquad k = 6$$

b. $F(5/6, 1/2) \Rightarrow$ use 2 integrals

$$\int_0^{1/2}\int_0^x 6(x-y)\,dy\,dx + \int_{1/2}^{5/6}\int_0^{1/2} 6(x-y)\,dy\,dx = \overset{1/8}{.125} + \overset{5/12}{.4167}$$

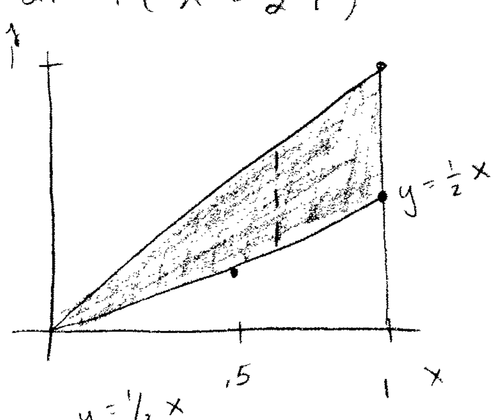
$$= .5417 \quad \left(\frac{13}{24}\right)$$

c. $P(X \geq 3/4,\ Y \geq 1/4)$



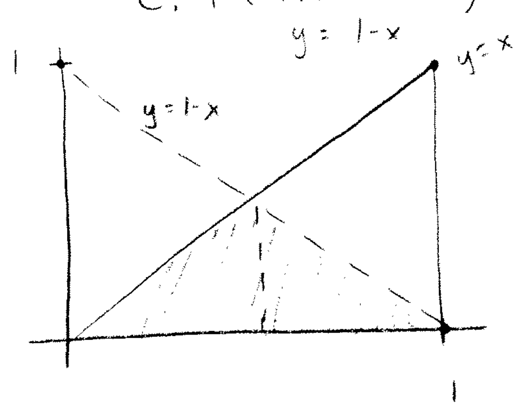
$$\int_{.75}^1 \int_{.25}^x 6(x-y)\,dy\,dx = .296875$$

d. $P(X \geq 2Y)$



$$\int_0^1 \int_{1/2 x}^x 6(x-y)\,dy\,dx = .25$$

e. $P(X+Y \leq 1)$

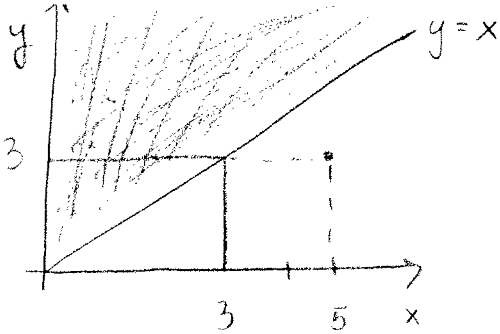


use 2 integrals

$$\int_0^{1/2}\int_0^x 6(x-y)\,dy\,dx + \int_{1/2}^1\int_0^{1-x} 6(x-y)\,dy\,dx$$

$$= .125 + .375 = .5$$

(from above)

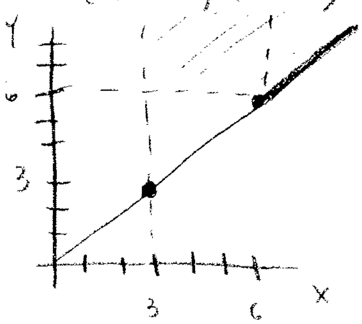
2. 

a. $F(5,3)$ Well since $x<y$, we can't accrue any + prob. beyond $F(3,3)$, so $F(5,3) = F(3,3)$

$$\int_0^3 \int_x^3 2e^{-(x+y)}\,dy\,dx = .902905$$

$$\left(\frac{(e^3-1)^2}{e^6}\right)$$

b. $P(X>3, Y>6)$

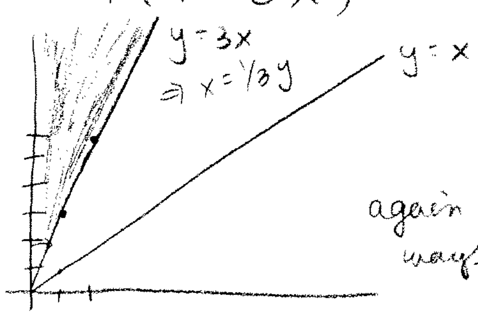


use 2 integrals in x-outer or slice y and get

$$\int_6^\infty \int_3^y 2e^{-(x+y)}\,dx\,dy = \frac{2e^3-1}{e^{12}} \approx .00024067\text{5}$$

very low

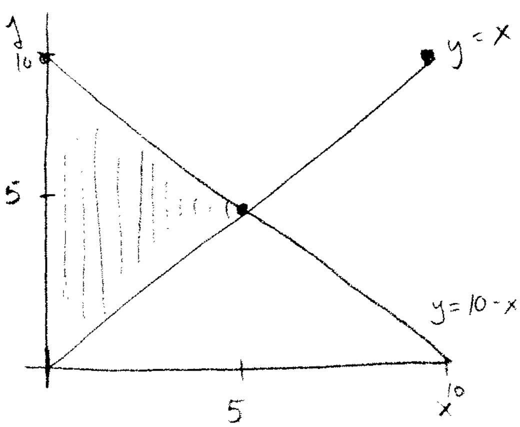
c. $P(Y>3X)$



$$\int_0^\infty \int_0^{\frac{1}{3}y} 2e^{-(x+y)}\,dx\,dy = \frac{1}{2}$$

again 2 ways to do

d. $P(X+Y \le 10)$



$$\int_0^5 \int_x^{10-x} 2e^{-(x+y)}\,dy\,dx = 1 - \frac{11}{e^{10}} = .999501$$

3. $X \sim Uni(0,1)$ $\quad f(x) = 1$ for $x \in [0,1]$

$Y|X \sim Uni(0, x^2)$ $\quad f(y|x) = \frac{1}{x^2}$, $y \in [0, x^2]$

$$f(x,y) = f_X(x)\, f(y|x) = 1 \cdot \frac{1}{x^2} = \frac{1}{x^2} \text{ for } 0 \le y < x^2,\ x \in [0,1]$$

0, o.w.

4. a.

$$f_X(x) = \int_Y f(x,y)\,dy = \int_0^x 6(x-y)\,dy = 3x^2, \quad 0 \le x \le 1$$
$$0, \text{ o.w.}$$

b.

$$f_Y(y) = \int_X f(x,y)\,dx = \int_0^y 2e^{-(x+y)}\,dx = 2e^{-2y}(e^y - 1)$$

c.

$$f(y|x) = \frac{6(x-y)}{3x^2} \quad 0 \le y \le x$$

when $x = 2/3$, this is

$$f(y|x=2/3) = \frac{\cancel{6}^{3}(2/3 - y)}{\cancel{3}(\cancel{2}/\cancel{3})^{\cancel{2}}} = \frac{4-6y}{4/3}$$
$$= \frac{12-18y}{4} = 3 - \frac{9}{2}y$$

$$P(Y \le 1/2 \mid X = 2/3) = \int_0^{1/2} 3 - \frac{9}{2}y\,dy$$
$$= .9375$$

d.

$$f(x|y) = \frac{\cancel{2}e^{-(x+y)}}{\cancel{2}e^{-2y}(e^y-1)} = \frac{e^{-(x+y)}}{e^{-2y}(e^y-1)}, \quad \text{for } 0 < x < y < \infty$$

when $y = 4$ this is

$$\frac{e^{-(x+4)}}{e^{-8}(e^4-1)}, \quad 0 < x < 4$$

$$P(X > 1 \mid Y = 4) = \int_1^4 \frac{e^{-(x+4)}}{e^{-8}(e^4-1)}\,dx = \int_1^4 \frac{e^8 e^{-(x+4)}}{e^4-1}\,dx$$

Note $x < y$

$$= \int_1^4 \frac{e^{4-x}}{e^4-1}\,dx = \frac{e^3-1}{e^4-1} = .356086$$