Solutions

Problem Session 4 for Math 29: Gamma Rays and Normal Models

1. Finding an MGF
Suppose X is a random variable with pdf $f(x) = (1/8)x, 0 \leq x \leq 4$, and 0, otherwise.
a. Denote the mgf of X by $M_X(t)$. If Y = 3X+4, what is the mgf of Y in terms of $M_X(t)$?
b. Find the mgf of X (hint: integration by parts).
c. If you had to find E(X) here, does it appear to be easier to find it using moments (taking derivatives of the mgf) or the original way we defined expectation?

a. $M_Y(t) = e^{tb} M_X(at) = e^{4b} M_X(3t)$

b. $M_X(t) = E(e^{tx}) = \int_0^4 \frac{x}{8} e^{tx}\,dx = \frac{1}{8}\int_0^4 xe^{tx}\,dx$

$u = x \quad dv = e^{tx}dx$
$du = dx \quad v = \frac{1}{t}e^{tx}$

$= \frac{1}{8}\left[\frac{x}{t}e^{tx}\Big|_0^4 - \frac{1}{t}\int_0^4 e^{tx}\,dx\right] = \frac{4}{8t}e^{4t} - \frac{1}{8t}\cdot\frac{1}{t}e^{tx}\Big|_0^4$

$= \frac{1}{8t}\left[4e^{4t} - \frac{e^{4t}}{t} + \frac{1}{t}\right]$ or $\frac{1}{8t}\left[4e^{4t} + \frac{1-e^{4t}}{t}\right]$

c. Easier doing directly.

$E(X) = \int_0^4 \frac{1}{8}x^2\,dx = \frac{x^3}{24}\Big|_0^4 = \frac{8}{3}$

2. The pH of the red sludge from the Hungarian alumina plant is of interest because of the quantity of sludge released into the rivers nearby. The pH has been decreasing from initial values near 13, but pH levels at 9 or above are still high, and indicate further efforts are needed to reduce pH (dumping plaster or absorbing chemicals). Suppose the probability of obtaining a high pH (9 or above) is .65. An EU official has commissioned a pH study in which a total of 248 sludge samples are taken (you can assume this is a random sample, and treat samples as independent) a week after the flood. What is the approximate probability that more than 172 samples have high pH?

X = # samples with high pH is $\sim Bin(248, .65)$

Try Normal Approx. to Binomial $\quad np = 248(.65) = 161.2 \quad n(1-p) = 248(.35) = 86.8 \quad \geq 10$?

Then X is $\approx \sim N(161.2, 7.5113)$

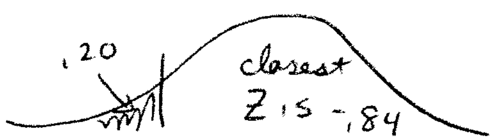
$P(X > 172) = P(X \geq 173) \underset{\text{use cc}}{\cong} P(X \geq 172.5)$

$= P\left(Z \geq \frac{172.5 - 161.2}{7.5113}\right) = P(Z \geq 1.5044) = P(Z \geq 1.50)$

$= .0668$

3. Average human body temperature is often reported as 98.6 degrees Fahrenheit. It's actually closer to 98.2 as an average. Assuming the standard deviation is .7 degrees, and that human body temperature can be modelled using a normal distribution (it's actually not a bad fitting model), address the following questions.
a. The coolest 20 percent of people have body temperatures at or below what temperature?
b. If a fever is 101 degrees or above, how likely is someone to have a fever temperature as their normal body temperature?
c. What percentage of people have body temperatures below 98 degrees F?

Let X be human body temp $\quad X \sim N(98.2, .7)$

a. 

$X = \mu + \sigma Z = 98.2 + .7(-.84)$

$=$

b. $P(X \geq 101) = P\left(Z \geq \frac{101 - 98.2}{.7}\right) = P(Z \geq 4) \approx 0$ to 4 decima[ls]

very unlikely

c. $P(X \leq 98) = P\left(Z \leq \frac{98 - 98.2}{.7}\right) = P(Z \leq -.2857)$

$= P(Z \leq -.29) = .3859 \qquad 38.59\%$

only 2 decimals on table

4. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density given by

$$f(y) = cy^2(1-y)^4,\ 0 \le y \le 1,$$

and 0, otherwise. What value of c makes this a valid pdf? What is the expected proportion of time per day that all counters are busy? What distribution does Y have? (Be specific, name and parameters.)

Y is $Beta(\alpha = 3, \beta = 5) \Rightarrow$

$$c \text{ must be } \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(8)}{\Gamma(3)\Gamma(5)} = \frac{7!}{2!\,4!} = \frac{7\cdot 6\cdot 5}{2} = 105$$

$$E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{3}{8} = .375$$

5. The response times for an online computer terminal have approximately a gamma distribution with mean 4 seconds and variance 8 seconds^2. Identify the parameters of the gamma distribution. Determine the probability the response time is longer than 10 seconds (Setup only, do not evaluate).

$Y \sim Gamma(\alpha, \beta)$ has $E(Y) = \alpha\beta$ and $V(Y) = \alpha\beta^2$

$\alpha\beta = 4 \qquad \alpha\beta^2 = 8 \qquad \beta = 2$ and $\alpha = 2$

Y is $Gamma(2, 2)$

$$f(y) = \begin{cases} \frac{1}{1\cdot 2^2} x^1 e^{-x/2}, & x \ge 0 \\ 0, & o.w. \end{cases}$$

$$P(Y > 10) = \int_{10}^{\infty} \frac{1}{4} x e^{-x/2}\, dx$$

$$\text{or} \quad 1 - \int_0^{10} \frac{1}{4} x e^{-x/2}\, dx$$

6. The length of time Y necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hours. The formula $C = 100 + 40Y + 3Y^2$ relates the cost C to the square of the time to completion. What are the mean and variance of C?

$Y \sim Exp(10) \qquad E(Y) = 10 \qquad V(Y) = 100 \qquad E(Y^2) = 200 = V(Y) + [E(Y)]^2$

$$E(C) = E(100 + 40Y + 3Y^2) = 100 + 40E(Y) + 3E(Y^2)$$
$$= 100 + 40(10) + 3(200) = 1100$$

$V(C) = E(C^2) - [E(C)]^2 =$ go find $E(C^2)$ and then finish

$$E(C^2) = E(100^2 + 100 \cdot 40Y + 100 \cdot 3Y^2 + 100 \cdot 40Y + 40^2 Y^2 + 40 \cdot 3 \cdot Y^3 + 100 \cdot 3Y^2 + 40 \cdot 3 \cdot Y^3 + 9Y^4)$$
$$= 100^2 + 4000E(Y) + 300E(Y^2) + 4000E(Y) + 1600E(Y^2) + 120E(Y^3) + 300E(Y^2) + 120E(Y^3) + 9E(Y^4)$$
$$= 100^2 + 8000E(Y) + 2200E(Y^2) + 240E(Y^3) + 9E(Y^4)$$
$$= 100^2 + 8000(10) + 2200(200) + 240(6000) + 9(24000) = 2186000$$

Using same logic as from class for Gamma Exp value, (Recall $\alpha = 1 \Rightarrow Exp$)

$E(Y) = \alpha\beta$

$E(Y^2) = \alpha(\alpha+1)\beta^2$ i.e. $E(Y^k) = \alpha(\alpha+1)\cdots(\alpha+k-1)\beta^k$

$E(Y^3) = \alpha(\alpha+1)(\alpha+2)\beta^3$

$E(Y^4) = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4$

In this example $\alpha = 1$, $\beta = 10$ so...

$E(Y) = 10$, $E(Y^2) = 200$, $E(Y^3) = 6000$, $E(Y^4) = 24000$

So! $V(C) = E(C^2) - [E(C)]^2 = 2186000 - (1100)^2 = 976000$