

Problem Session 6 for Math 29: Xforms and Convergence

Read through the problems and choose the ones you feel would most benefit you to work on (i.e. the ones you are least comfortable tackling). Feel free to ask for my help as usual.

1. Let $X_1, X_2, \dots, X_n$ be iid exponential RVs with common mean β .
- a. Show that $X_{(1)}$ (minimum) has an exponential distribution with mean $\frac{\beta}{n}$.
- b. Set up an integral to find $P(X_{(1)} < 4)$ when $n=8$ and $\beta = 3$.

a. By formulas for $X_{(1)}$,

$$g_1(x) = n [1 - F(x)]^{n-1} f(x).$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, x \geq 0$$

0, o.w.

$$F(x) = 1 - e^{-x/\beta}, x \geq 0$$

0, $x < 0$

$$\Rightarrow g_1(x) = n [1 - (1 - e^{-x/\beta})]^{n-1} \cdot \frac{1}{\beta} e^{-x/\beta}$$

$$= n [e^{-x/\beta}]^{n-1} \cdot \frac{1}{\beta} e^{-x/\beta} \Rightarrow \frac{n}{\beta} e^{-nx/\beta}$$

$$= \frac{1}{\beta/n} e^{-x/(\beta/n)}$$

$$\sim \text{Exp}\left(\frac{\beta}{n}\right) \text{ pdf.}$$

b. $X_{(1)} \sim \text{Exp}\left(\frac{3}{8}\right)$

$$P(X_{(1)} < 4) = \int_0^4 \frac{8}{3} e^{-8x/3} dx = .999977$$

2. An anthropologist wants to estimate the average height of men for a certain race of people. If the population standard deviation of male heights is assumed to be 2.5 inches, and 100 men are randomly sampled, what is the probability that the difference between the sample mean and true population mean will not exceed .5 inches? What result did you use to help compute the probability?

$$\mu = \text{unknown} \quad \sigma = 2.5 \quad n = 100$$

$$P(|\bar{X} - \mu| \leq .5) = P(-.5 \leq \bar{X} - \mu \leq .5)$$

$$\text{Using } \underline{\underline{\text{CLT}}}, \quad Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \Rightarrow N(0,1), \text{ so}$$

$$P\left(\frac{-0.5}{2.5/\sqrt{100}} \leq Z \leq \frac{0.5}{2.5/\sqrt{100}}\right) = P(-2 \leq Z \leq 2)$$

$$= 1 - 2(.0228) = .9544$$

No CC assuming height is continuous.

3. Playing with MGFs.

a. Let $X_1, X_2, \dots, X_n$ be a sample of n Poisson random variables with possibly different λ , on the same time scale. Using the method of mgfs, what is the distribution of $Y = \sum_{i=1}^n X_i$? The mgf for a Poisson (λ) is $\exp[\lambda(e^t - 1)]$.

b. Suppose you consider 2 independent electrical components which each have life length governed by an exponential distribution with mean 1. Use the method of mgfs to find the mgf and then density function for the average life length of the two components. The mgf for an exponential with parameter θ is $(1 - \theta t)^{-1}$.

$$a. M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n e^{\lambda_i(e^t - 1)} = e^{(e^t - 1) \sum_{i=1}^n \lambda_i}$$

$$\text{So, } Y = \sum_{i=1}^n X_i \text{ is Poisson}(\sum \lambda_i)$$

$$b. X_1 \perp X_2 \quad \text{Exp}(1) \quad \bar{X}_2 = \frac{X_1 + X_2}{2}$$

$$Y = X_1 + X_2 \text{ is Gamma}(2, 1) \quad \text{mgf}$$

$$f_Y(y) = y e^{-y}, \quad y \geq 0 \quad (1-t)^{-2}$$

$$\bar{X}_2 = \frac{1}{2} Y \quad \text{can do x forms or mgfs}$$

$$Y = 2 \bar{X}_2$$

$$M_{\bar{X}_2} = M_Y\left(\frac{1}{2}t\right) \quad M_{\bar{X}_2} = \left(1 - \frac{1}{2}t\right)^{-2}$$

$$f_{\bar{X}_2} = 2 \cdot f_Y(2\bar{X}_2) = 4\bar{X}_2 e^{-2\bar{X}_2}, \quad \bar{X}_2 \geq 0$$

$$\bar{X}_2 \sim \text{Gamma}(2, 1/2)$$

4. Suppose that the distribution of a number of defects on any given bolt of cloth is a Poisson distribution with mean 5. Suppose a random sample of 125 bolts of cloth are examined and the number of defects are counted.

a. Using an approximation result, what is the probability the total number of defects counted is less than 687.5? (Hint: look at 3a. for the distribution of the sum of Poissons).

b. Determine the probability that the average number of defects per bolt in the sample of 125 bolts will be less than 5.5.

c. What do you notice about the probabilities in a and b? Note: something "nice" should occur if you are consistent in your use of continuity corrections.

a. Total defects $\approx \text{Poisson}(5 \cdot 125 = 625) = Y$

$$P(Y < 687.5) = P(Y \leq 687)$$

Normal $\approx$ to Poisson

$$\stackrel{cc}{\approx} P(Y < 687.5) = P(Z < \frac{687.5 - 625}{\sqrt{625}})$$

$$Z = \frac{Y - 625}{\sqrt{625}} \sim N(0,1)$$

$$= P(Z \leq 2.5) = .9938$$

b. $\bar{X} \sim N(5, \frac{\sqrt{5}}{\sqrt{125}} = .2)$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 5}{\sqrt{5}/\sqrt{125}}$$

$$P(\bar{X} < 5.5) = P(Z < \frac{5.5 - 5}{.2})$$

$\bar{X}$ is discrete, so
cc only in a for total

$$= P(Z < 2.5) = .9938$$

c. same probability

$$\begin{array}{ccc} 5.5(125) & = & 687.5 \\ \uparrow & & \uparrow \\ \text{avg.} & & \text{total} \end{array}$$

5. Weibull density function.

The Weibull density function is given by

$$f(x) = \frac{1}{\alpha} m x^{m-1} e^{-x^m/\alpha}, x > 0, \text{ and } 0, \text{ o.w.}$$

where α and m are positive constants.

Let Y be an exponential RV with mean θ . What distribution does $W = \sqrt{Y}$ have? Hint: The Weibull info should be useful to you somewhere.

$$Y \sim \text{Exp}(\theta) \quad f_Y = \frac{1}{\theta} e^{-y/\theta}, y > 0, \text{ o.w.}$$

$$W = \sqrt{Y} \quad Y = W^2 \quad y > 0, w > 0$$

$$f_W = 2w f_Y(w^2)$$

$$= \frac{2w}{\theta} e^{-w^2/\theta}, w > 0 \quad \sim \text{Weibull}(\alpha = \theta, m = 2)$$

6. Suppose Y_1 and Y_2 denote the length of life of two different electrical components in hours and that they have a joint continuous distribution given by $f(y_1, y_2) = (1/8)y_1 \exp(-(y_1 + y_2)/2)$, where $y_1 > 0, y_2 > 0$.

a. The relative efficiency of the two components is given by $U = Y_2/Y_1$. Suppose we want to find the marginal pdf of U . Letting $V = Y_1$, what is the joint pdf of U and V ?

b. How would you find the marginal of U from the joint in a.? (Set it up). Using integration by parts (twice), the marginal should be $g(u) = 2/(1+u)^3, u > 0$, and 0, otherwise, if you decide to check this later.

$$a. \quad u = \frac{Y_2}{Y_1} \quad V = Y_1 \quad \Rightarrow \quad Y_1 = V = h_1(u, v) \quad Y_2 = uV = h_2(u, v)$$

$$J = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v \quad |J| = v$$

$$\begin{aligned} v &> 0 \quad \text{b/c } y_1 > 0 \\ uv &> 0 \quad \text{b/c } y_2 > 0 \\ &\Rightarrow u > 0 \end{aligned}$$

$$f_{uv} = f_{Y_1, Y_2}(v, uv) \cdot v$$

$$= \frac{v^2}{8} \exp \left\{ -\frac{1}{2} (v + uv) \right\}$$

$$b. \quad f_u = \int_0^{\infty} \frac{v^2}{8} \exp \left\{ -\frac{1}{2} v(u+1) \right\} dv$$