

$$\begin{aligned}
 \Pr(W > M) &= \Pr(W - M > 0) \\
 &= \Pr\left(Z > \frac{3}{5^{1/2}}\right) = \Pr(Z > 1.342) \\
 &= 1 - \Phi(1.342) = 0.090.
 \end{aligned}$$

Thus, the probability that the woman will be taller than the man is 0.090. ◀

The Lognormal Distribution

It is very common to use normal distributions to model logarithms of random variables. For this reason, a name is given to the distribution of the original random variables before transforming. If $\log(X)$ has a normal distribution with mean μ and variance σ^2 , we say that X has a lognormal distribution with parameters μ and σ^2 .

Example 5.6.7 Failure Times of Ball Bearings. Products that are subject to wear and tear are generally tested for endurance in order to estimate their useful lifetimes. Lawless (1982, Example 5.2.2) describes data taken from Lieblein and Zelen (1956), which are measurements of the numbers of millions of revolutions before failure for 23 ball bearings. The lognormal distribution is one popular model for times until failure. Figure 5.4 shows a histogram of the 23 lifetimes together with a lognormal p.d.f. with parameters chosen to match the observed data. The bars of the histogram in Fig. 5.4 have a similar interpretation to those in Fig. 5.1, as described in Example 5.6.1. Suppose that the engineers are interested in knowing how long to wait until there is a 90 percent chance that a ball bearing will have failed. Then they want the 0.9 quantile of the distribution of lifetimes. Let X be the time to failure of a ball bearing. The lognormal distribution of X plotted in Fig. 5.4 has parameters 4.15 and 0.5334^2 . The d.f. of X would then be $F(x) = \Phi([\log(x) - 4.15]/0.5334)$, and the quantile function would be

$$F^{-1}(p) = e^{4.15 + 0.5334\Phi^{-1}(p)},$$

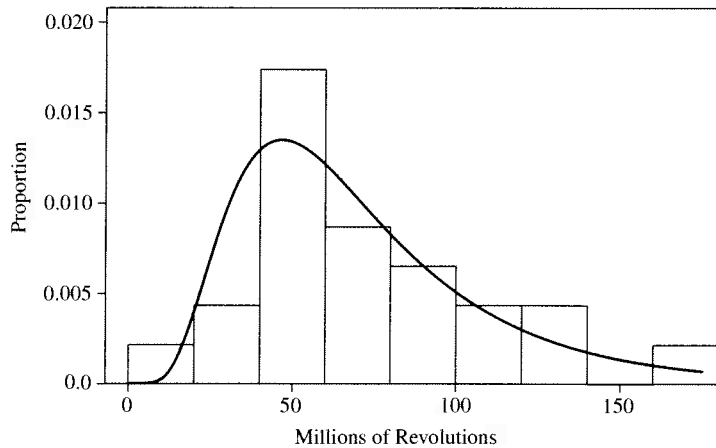


Figure 5.4 Histogram of lifetimes of ball bearings and fitted lognormal p.d.f. for Example 5.6.7.

where Φ^{-1} is the quantile function of the standard normal distribution. With $p = 0.9$, we get $\Phi^{-1}(0.9) = 1.28$ and $F^{-1}(0.9) = 125.6$.

The moments of a lognormal random variable are easy to compute based on the m.g.f. of a normal distribution. If $Y = \log(X)$ has a normal distribution with mean μ and variance σ^2 , then the m.g.f. of Y is $\psi(t) = \exp(\mu t + 0.5\sigma^2 t^2)$. But, the definition of ψ is $\psi(t) = E(e^{tY})$. Since $Y = \log(X)$, we have

$$\psi(t) = E(e^{tY}) = E(e^{t \log(X)}) = E(X^t).$$

It follows that $E(X^t) = \psi(t)$ for all real t . In particular, the mean and variance of X are

$$\begin{aligned} E(X) &= \psi(1) = \exp(\mu + 0.5\sigma^2), \\ \text{Var}(X) &= \psi(2) - \psi(1)^2 = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]. \end{aligned} \quad (5.6.9)$$

Example 5.6.8 Stock and Option Prices. Consider a stock like the one in Example 5.6.2 whose current price is S_0 . Suppose that the price at u time units in the future is $S_u = S_0 e^{Z_u}$, where Z_u has a normal distribution with mean μu and variance $\sigma^2 u$. Note that $S_0 e^{Z_u} = e^{Z_u + \log(S_0)}$ and $Z_u + \log(S_0)$ has a normal distribution with mean $\mu u + \log(S_0)$ and variance $\sigma^2 u$. So S_u has a lognormal distribution with parameters $\mu u + \log(S_0)$ and $\sigma^2 u$.

Black and Scholes (1973) developed a pricing scheme for options on stocks whose prices follow a lognormal distribution. For the remainder of this example, we shall consider a single time u and write the stock price as $S_u = S_0 e^{\mu u + \sigma u^{1/2} Z}$, where Z has a standard normal distribution. Suppose that we need to price the option to buy one share of the above stock for the price q at a particular time u in the future. As in Example 4.1.4 on page 186, we shall use risk-neutral pricing. That is, we force the present value of $E(S_u)$ to equal S_0 . If u is measured in years and the risk-free interest rate is r per year, then the present value of $E(S_u)$ is $e^{-ru} E(S_u)$. (This assumes that compounding of interest is done continuously instead of just once as it was in Example 4.1.4. The effect of continuous compounding is examined in Exercise 25.) But $E(S_u) = S_0 e^{\mu u + \sigma^2 u/2}$. Setting S_0 equal to $e^{-ru} S_0 e^{\mu u + \sigma^2 u/2}$ yields $\mu = r - \sigma^2/2$ when doing risk-neutral pricing.

Now, we can determine a price for the specified option. The value of the option at time u will be $h(S_u)$, where

$$h(s) = \begin{cases} s - q & \text{if } s > q, \\ 0 & \text{otherwise.} \end{cases}$$

Now, set $\mu = r - \sigma^2/2$, and it is easy to see that $h(S_u) > 0$ if and only if

$$Z > \frac{\log\left(\frac{q}{S_0}\right) - (r - \sigma^2/2)u}{\sigma u^{1/2}}. \quad (5.6.10)$$

We shall refer to the constant on the right-hand side of Eq. (5.6.10) as c . The risk-neutral price of the option is the present value of $E(h(S_u))$, which equals

$$e^{-ru} E[h(S_u)] = e^{-ru} \int_c^\infty \left[S_0 e^{[r - \sigma^2/2]u + \sigma u^{1/2} z} - q \right] \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} dz. \quad (5.6.11)$$

To compute the integral in Eq. (5.6.11), split the integrand into two parts at the $-q$. The second integral is then just a constant times the integral of a normal p.d.f., namely

$$-e^{-ru}q \int_c^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} dz = -e^{-ru}q[1 - \Phi(c)].$$

The first integral in Eq. (5.6.11), is

$$e^{-\sigma^2 u/2} S_0 \int_c^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2 + \sigma u^{1/2} z} dz.$$

This can be converted into the integral of a normal p.d.f. times a constant by completing the square (see Exercise 24). The result of completing the square is

$$e^{-\sigma^2 u/2} S_0 \int_c^\infty \frac{1}{(2\pi)^{1/2}} e^{-(z - \sigma u^{1/2})^2/2 + \sigma^2 u/2} dz = S_0 [1 - \Phi(c - \sigma u^{1/2})].$$

Finally, combine the two integrals into the option price, using the fact that $1 - \Phi(x) = \Phi(-x)$:

$$S_0 \Phi(\sigma u^{1/2} - c) - q e^{-ru} \Phi(-c). \quad (5.6.12)$$

This is the famous *Black-Scholes formula* for pricing options. As a simple example, suppose that $q = S_0$, $r = 0.06$ (six percent interest), $u = 1$ (one year wait), and $\sigma = 0.1$. Then (5.6.12) says that the option price should be $0.0746 S_0$. If the distribution of S_u is different from the form used here, simulation techniques (see Chapter 11) can be used to help price options. ◀

The p.d.f. of the lognormal distribution will be found in Exercise 17 of this section. The d.f. of the lognormal distribution is easily constructed from the standard normal d.f. Φ . Let X have a lognormal distribution with parameters μ and σ^2 . Then

$$\Pr(X \leq x) = \Pr(\log(X) \leq \log(x)) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right).$$

The results from earlier in this section about linear combinations of normal random variables translate into results about products of powers of lognormal random variables. Results about sums of independent normal random variables translate into results about products of independent lognormal random variables.

Summary

We introduced the family of normal distributions. The parameters of each normal distribution are its mean and variance. A linear combination of independent normal random variables has a normal distribution with mean equal to the linear combination of the means and variance determined by Corollary 4.3.1. In particular, if X has a normal distribution with mean μ and variance σ^2 , then $(X - \mu)/\sigma$ has a standard normal distribution (mean 0 and variance 1). Probabilities and quantiles for normal distributions can be obtained from tables or computer programs for standard normal probabilities and quantiles. For example, if X has a normal distribution with mean μ and variance σ^2 , then the d.f. of X is $F(x) = \Phi([x - \mu]/\sigma)$ and the quantile function of X is $F^{-1}(p) = \mu + \Phi^{-1}(p)\sigma$, where Φ is the standard normal d.f.