Name: Detailed Solutions

Math 29 – Probability

Second Midterm Exam

Instructions:

1. Show all work. You may receive partial credit for partially completed problems.
2. You may use calculators and a one-sided sheet of reference notes. I have also provided various z-tables for your use. You may not use any other references or any texts.
3. You may not discuss the exam with anyone but me.
4. Suggestion: Read all questions before beginning and complete the ones you know best first. Point values per problem are displayed below if that helps you allocate your time among problems. Problems 1 and 2 are spread out over several pages so you have lots of space to work.
5. You need to demonstrate that you can solve all integrals that do not have a (DO NOT SOLVE) statement. I.E. Do the integrals by hand and write out some work showing how you solved the integration, including if necessary integration by parts.
6. Good luck!

Problem	1	2	3	Total
Points Earned				
Possible Points	15	25	10	50

1. You may have heard that there are relationships between music and math. In this problem, we consider a very specific relationship between music and math. Suppose you are going to compose a music piece and you decide to construct it using a Markov Chain (you can actually build a music composition this way). For this example, you can only play three notes - A, C sharp, and E flat. The partial transition matrix is given at right (current state = row, next state = column).

	A	C sharp	E flat
A	.1	.6	.3
C sharp	.25	.05	.7
E flat	.7	.3	0

a. Complete the transition matrix so that it is a valid transition matrix for a Markov Chain. You will use the completed transition matrix for all parts below.

row sums must be 1

b. Are there any absorbing sets apart from the entire set {A, C sharp, E flat}? If yes, state them. Is this transition matrix reducible or irreducible?

There are no absorbing sets apart from S.

Hence the matrix is irreducible.

c. What is the probability A is the note that is played 2 notes from now assuming you just played E flat?

P(A in 2 notes | E flat current)

$$= \overset{.7}{P_{EA}}\,\overset{.1}{P_{AA}} + \overset{.3}{P_{EC}}\,\overset{.25}{P_{CA}} + \overset{0}{P_{EE}}\,\overset{.7}{P_{EA}}$$

$$= .07 + .075 = .145$$

d. If you are equally likely to start with any note, what is the probability A is the third note played? (I.E. you play an initial note, one more note, and then A) ← ONLY 3 notes are played

You can use c to help here!

$$P(A \text{ is } 3^{rd} \text{ note played}) = \underset{P(\text{Start Eb})}{\tfrac{1}{3}}\left(\text{Play A} \mid \text{Start Eb}\right) + \underset{P(\text{Start A})}{\tfrac{1}{3}}\left(\text{Play A} \mid \text{start A}\right)$$

$$+ \underset{P(\text{start C\#})}{\tfrac{1}{3}}\left(\text{Play A} \mid \text{start C\#}\right)$$

from c!

$$= \tfrac{1}{3}(.145) + \tfrac{1}{3}\left(\overset{.1}{P_{AA}}\,\overset{.1}{P_{AA}} + \overset{.6}{P_{AC}}\,\overset{.25}{P_{CA}} + \overset{.3}{P_{AE}}\,\overset{.7}{P_{EA}}\right)$$

$$+ \tfrac{1}{3}\left(\overset{.25}{P_{CA}}\,\overset{.1}{P_{AA}} + \overset{.05}{P_{CC}}\,\overset{.25}{P_{CA}} + \overset{.7}{P_{CE}}\,\overset{.7}{P_{EA}}\right) = \tfrac{1}{3}(.145 + .37 + .527$$

$$= \tfrac{1}{3}(1.0425)$$

$$= .3475$$

You could solve by getting the 2 step matrix, but only 1 of you did it that way.

1. continued.

e. In the long-run (assuming an equilibrium distribution exists) what fraction of notes will be C sharps?

Use the fact that one equation involves a 0!

$$\pi = \pi P \implies [\pi_1\ \pi_2\ \pi_3] = [\pi_1\ \pi_2\ \pi_3] \begin{array}{c} \begin{array}{ccc} A & C\# & Eb \end{array} \\ \begin{bmatrix} .1 & .6 & .3 \\ .25 & .05 & .7 \\ .7 & .3 & 0 \end{bmatrix} \end{array}$$

row x col

Equations:

1 $\pi_1 = .1\pi_1 + .25\pi_2 + .7\pi_3$

2 $\pi_2 = .6\pi_1 + .05\pi_2 + .3\pi_3$

3 $\pi_3 = .3\pi_1 + .7\pi_2$

4 $\pi_1 + \pi_2 + \pi_3 = 1$

use 3 in 1 $\Rightarrow$

$\pi_1 = .1\pi_1 + .25\pi_2 + .7(.3\pi_1 + .7\pi_2)$

$.69\pi_1 = .74\pi_2$

$\pi_1 = \frac{74}{69}\pi_2$

$= 1.072464\,\pi_2$

$\Rightarrow \pi_3 = 1.021739\,\pi_2$ plugging back in 3.

Now, using 4, sub in for π_1 & π_3 all in terms of π_2

$1 = (3.094203)\pi_2$

$\pi_2 = .323185$

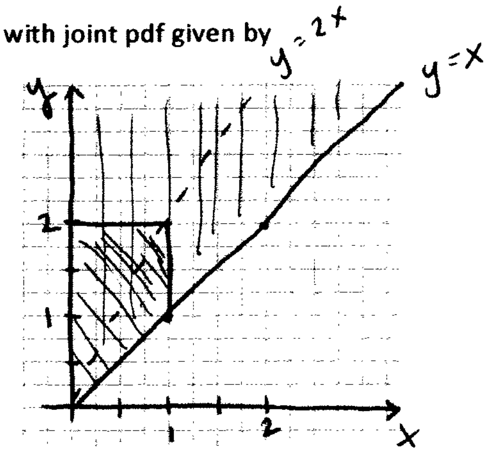
The fraction of notes that will be C#'s is 32.32%.

Note: It didn't ask for the entire equilibrium distribution.

If you did row x row, you got exactly 1/3 as the fraction.

2. Suppose X and Y are jointly distributed continuous random variables with joint pdf given by

$f(x,y) = \lambda^2 e^{-\lambda y}$, for $0 \le x < y < \infty$, and where $\lambda > 0$.



a. Sketch and shade the region where the joint pdf takes positive density on the graph paper at right.

b. Set up an integral or integrals (DO NOT SOLVE) to find P(Y > 2X).

$$\int_0^\infty \int_{2x}^\infty \lambda^2 e^{-\lambda y}\, dy\, dx \qquad \text{OR}$$

$$\int_0^\infty \int_0^{y/2} \lambda^2 e^{-\lambda y}\, dx\, dy$$

c. Find F(1,2). You will need to leave your answer in terms of λ.

shape

either takes 2 integrals if inner is x or 1 if inner is y.

$$F(1,2) = P(X \le 1, Y \le 2) = \int_0^1 \int_x^2 \lambda^2 e^{-\lambda y}\, dy\, dx$$

$$= \int_0^1 -\lambda e^{-\lambda y}\Big|_x^2\, dx = \int_0^1 \left(\lambda e^{-\lambda x} - \lambda e^{-2\lambda}\right) dx = \left(-e^{-\lambda x} - x\lambda e^{-2\lambda}\right)\Big|_0^1$$

$$= 1 - e^{-\lambda} - \lambda e^{-2\lambda}$$

Setup using x on inside...

Answer same

$$\underbrace{\int_0^1 \int_0^y \lambda^2 e^{-\lambda y}\, dx\, dy}_{\text{bottom triangle}} + \underbrace{\int_1^2 \int_0^1 \lambda^2 e^{-\lambda y}\, dx\, dy}_{\text{square}} = 1 - e^{-\lambda} - \lambda e^{-2\lambda}$$

d. Find the marginal distribution of Y.

Integrate jt pdf over all values of $\underline{X}$ to get marginal of y.

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\, dx = x\, \lambda^2 e^{-\lambda y}\Big|_0^y$$

$$= \begin{cases} \lambda^2 y e^{-\lambda y}, & 0 < y < \infty \\ 0, & \text{o.w.} \end{cases}$$

Note: I did NOT grade the bounds, but this is what they are. $\longrightarrow$

2. continued.

e. Use the marginal distribution of Y to find the MGF of Y (i.e. solve as an expectation of a function of Y) when $\lambda = 1$.

From d, $f_Y(y) = \lambda^2 y e^{-\lambda y}$, $0 < y < \infty$; 0, o.w. $\xrightarrow{\lambda = 1}$ $y e^{-y}$, $0 < y < \infty$; 0, o.w.

$$E(e^{tY}) = \int_0^\infty e^{ty}(ye^{-y})\,dy$$

this is a correct setup

Solving...

$$= \int_0^\infty y\, e^{-y(1-t)}\,dy$$

modified Gamma Fn! $\alpha = 2$ $\beta = \frac{1}{1-t}$

$$= \Gamma(\alpha)\beta^\alpha = \Gamma(2)\left(\frac{1}{1-t}\right)^2 = (1-t)^{-2} = M_Y(t)$$

If you get stuck, pick something so you have it to use!

f. What is the distribution of Y when $\lambda = 1$? Justify your answer in one sentence.

$Y \sim$ Gamma $(\alpha = 2, \beta = 1)$ b/c the mgf in e. is a Gamma mgf and mgfs are unique!

g. Give the general conditional distribution of X given Y for a general λ, and then use it to find P(X>.5 | Y=2).

$$f_{X|Y} = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y}, \quad 0 \le x < y; \quad 0, \text{ o.w.}$$

Again I didn't grade on the bounds.

(You can notice X|Y is Uniform on (0, y)).

$$P(X > .5 \mid Y = 2) = \int_{.5}^{2} \frac{1}{2}\,dx = .5x\Big|_{.5}^{2} = \frac{1}{2}\left(\frac{3}{2}\right) = 3/4$$

Sidenote: This particular jt. dist is very nicely behaved. The marginal for X ends up being a dist we know also.

3. The number of particles emitted by a radioactive source is well-modeled by a Poisson (4 in 1 hour) distribution. $X \sim \text{Poisson}(4)$

a. What is the probability at most one particle is discharged from the source in a given hour?

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} = e^{-4} + 4e^{-4} = 5e^{-4} \approx .0916$$

b. Now suppose there are 5 independent radioactive sources. What is the probability exactly 2 of these sources discharge at most one particle in a given hour?

Y = # discharge @ most 1 $\quad Y \sim \text{Bin}(5, .0916)$

$$P(Y=2) = \binom{5}{2}(.0916)^2(.9084)^3 \approx .0629$$

c. Now suppose there are 500 independent radioactive sources. What is the approximate probability that at most 55 of these sources discharge at most one particle during a given hour? You should make your approximation as accurate as you can. ⇐ use continuity correction!

$Y \sim \text{Bin}(500, .0916)$

Check to use approx.

$np = 45.8$

$n(1-p) = 454.2$ $\geq 10.$ ✓

So, Y is $\approx N(45.8, 6.450172)$

$$P(Y \leq 55) \overset{CC}{\approx} P(Y \leq 55.5) = P\left(Z \leq \frac{55.5 - 45.8}{6.450172}\right)$$

$$= P(Z \leq 1.50) = .9332$$

d. Give a formula for $E(X^2)$ when X is Normal(μ, σ) in terms of the parameters of the distribution.

This was just to remind you of the quick trick:

$$V(X) = \sigma^2 = E(X^2) - \mu^2 \implies E(X^2) = \mu^2 + \sigma^2$$