Name: Solutions

Math 29 – Probability

**Practice** Third Midterm Exam 2

Instructions:

1. Show all work. You may receive partial credit for partially completed problems.
2. You may use calculators and a one-sided sheet of reference notes. You may not use any other references or any texts.
3. You may not discuss the exam with anyone but me.
4. Suggestion: Read all questions before beginning and complete the ones you know best first. Point values per problem are displayed below if that helps you allocate your time among problems.
5. You need to demonstrate that you can solve all integrals in problems that do not have a (DO NOT SOLVE) statement. I.E. write out some work showing how you solved the integration, including if necessary integration by parts.
6. Good luck!

| Problem | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points Earned | | | | | |
| Possible Points | 10 | 18 | 10 | 12 | 50 |

something like

1. A packaging company is trying to decide how many of each size box to produce for the holidays. They specialize in fairly small perfectly cube boxes. Let X be the length of a randomly generated cube (in inches) from their production line and suppose the pdf of X is given by (for current machine settings):

$$f(x) = \frac{1}{243}(x-3)^2, 3 \le x \le 12, \text{ and 0, otherwise.}$$

a. What is the pdf of $Y = X^3$, the volume of the generated cube box? Method of X forms in 1-D or CDFs

$Y = X^3 \quad X = Y^{1/3} = h(y) \quad h'(y) = \frac{1}{3} y^{-2/3}$

Bounds
$3 \le x \le 12$
$27 \le y \le 1728$

$$f_Y(y) = \frac{1}{3} y^{-2/3} f_X(Y^{1/3})$$

$$= \left(\frac{1}{3} y^{-2/3}\right) \frac{1}{243} \left(y^{1/3} - 3\right)^2, \quad 27 \le y \le 1728$$

$0$, o.w.

b. The executives want less than 60% of cube boxes produced to have a volume greater than 729 inches cubed. Are the executives going to be pleased with production? Justify your response with a probability-related calculation. Can do in relation to volume or length.

$Y = 729 \Rightarrow X = 9$

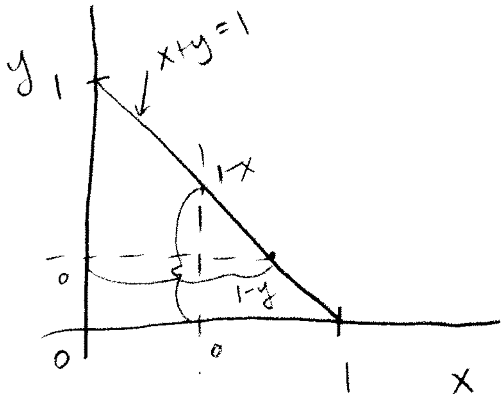
$$P(X \ge 9) = \int_9^{12} \frac{1}{243}(x-3)^2 \, dx \quad \begin{matrix} u = x-3 \\ du = dx \end{matrix} \quad = \int_6^9 \frac{1}{243} u^2 \, du$$

$$= \frac{1}{243} \cdot \frac{u^3}{3} \Big|_6^9 = \frac{1}{729}(729 - 216) = \frac{513}{729} = .7039 > 60\%$$

The executives will not be pleased with this production, b/c over 70% have vol. > 729 in.$^3$ not less than 60%.

2. Let X and Y be random variables such that $f_{X,Y}(x,y) = 24xy, 0 \le x, y, \le 1, 0 \le x+y \le 1$, and 0, ow.

a. Find Cov(X,Y).




$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Find these pieces, then combine.

$$E(XY) = \int_0^1\int_0^{1-y} 24x^2y^2\,dx\,dy = \int_0^1 8x^3y^2\Big|_0^{1-y} dy$$

$$= \int_0^1 8(1-y)^3y^2\,dy = \int_0^1 8y^2(-y^3+3y^2-3y+1)\,dy = \int_0^1 -8y^5+24y^4-24y^3+8y^2\,dy$$

$$= \frac{-8}{6}y^6 + \frac{24}{5}y^5 - 6y^4 + \frac{8}{3}y^3\Big|_0^1 = \frac{2}{15} = .133\overline{3}$$

$$E(X) = \int_0^1\int_0^{1-y} 24x^2y\,dx\,dy = \int_0^1 8x^3y\Big|_0^{1-y} dy = \int_0^1 8(1-y)^3y\,dy$$

$$= \int_0^1 -8y^4+24y^3-24y^2+8y\,dy = \frac{-8}{5}y^5+6y^4-8y^3+4y^2\Big|_0^1$$

$$= 2 - 8/5 = \frac{2}{5}$$

$$E(Y) = \int_0^1\int_0^{1-y} 24xy^2\,dx\,dy = \int_0^1 12x^2y^2\Big|_0^{1-y} dy = \int_0^1 12(1-y)^2y^2\,dy$$

$$= \int_0^1 12y^2(1-2y+y^2)\,dy = \int_0^1 12y^2-24y^3+12y^4\,dy =$$

$$4y^3 - 6y^4 + \frac{12}{5}y^5\Big|_0^1 = -2 + \frac{12}{5} = \frac{2}{5}$$

$\Rightarrow$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{2}{15} - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{-2}{75} = -.02\overline{6}$$

b. Are X and Y independent? Justify your answer with some work relating to marginal pdfs.

We know X and Y are NOT $\perp$ b/c their covariance is NOT 0, but we need to validate using marginal pdfs.

$X \perp Y \iff f_{XY} = f_X f_Y$

Show the jt. doesn't factor into product of the marginals.

$f_X = \int_0^{1-x} 24xy\,dy = 12xy^2\Big|_0^{1-x} = 12x(1-x)^2, \quad x \in (0,1)$

$X \sim Beta(2,3)$

$f_Y = \int_0^{1-y} 24xy\,dx = 12x^2y\Big|_0^{1-y} = 12y(1-y)^2, \quad y \in (0,1)$

$y \sim Beta(2,3)$ also

However,

$f_X(f_Y) = 144xy(1-x)^2(1-y)^2 \neq f_{XY} = 24xy,$

so X is NOT $\perp$ of Y.

3. A particular road construction project has restricted passage on the road to one lane, causing motorists to sometimes wait for traffic to move in the direction they are going. The time that a motorist spends waiting, X, is exponentially distributed with a mean of $\lambda$, where $\lambda$ is known to be normally distributed with a mean of $\mu$ and standard deviation of $\sigma$. $X|\lambda \sim Exp(\lambda)$ $\lambda \sim N(\mu, \sigma)$

a. Find the mean of the time a motorist spends waiting.

$$E(X) = E(E(X|\lambda)) = E(\lambda) = \mu$$

b. Find the variance of the time a motorist spends waiting.

$E(X|\lambda) = \lambda$

$V(X|\lambda) = \lambda^2$

$$V(X) = E(V(X|\lambda)) + V(E(X|\lambda))$$

$$= E(\lambda^2) + V(\lambda)$$

$$= \sigma^2 + \mu^2 + \sigma^2$$

$$= \mu^2 + 2\sigma^2$$

Note: $E(\lambda^2) = V(\lambda) + [E(\lambda)]^2 = \sigma^2 + \mu^2$

Use smoothing for both parts!

4. An infinite size "magical" holiday bowl of candy contains 10% Hershey miniature chocolate bars, 20% Hershey kisses, 30% Reese's miniature peanut butter cups, and the rest is non-chocolate candy. Suppose you are asked to randomly select a candy for each person in the class by drawing from the bag, so 30 pieces of candy.

Multinomial $(30, (.1, .2, .3, .4))$

a. How many miniature chocolate bars do you expect in your sample of 30 pieces?

$np_1 = .1(30) = 3$

b. What is the covariance between the number of peanut butter cups and number of kisses that you select in your sample of 30 pieces?

$Cov(X_3, X_2) = -np_3p_2 = -30(.3)(.2) = -1.8$

c. What is the approximate probability that there are fewer than 10 pieces of non-chocolate candy in your sample of 30 pieces? (Hint: use an approximation). You need to provide a numerical value as an answer.

$X_4 \sim Bin(30, .4)$ $\quad P(X_4 < 10) = P(X_4 \leq 9)$

Normal $\approx$ Binomial $\quad np = 12 \quad n(1-p) = 18 \geq 10$ ✓

$X_4$ is $\approx N(12, \sqrt{7.2} = 2.68)$

$P(X_4 \leq 9) \overset{cc}{\approx} P(X_4 \leq 9.5) = P\left(Z \leq \frac{9.5-12}{2.68}\right) = P(Z \leq -.93)$

$= .1762$

d. Suppose a second such magical bag of candy appears, and that you don't know its composition. If you wanted to estimate the proportion of chocolate candies in this new bag (p), you might count the number of chocolate candies (X) in a sample of size n. What can you say about the relationship between X/n and p in terms of convergence? What result allows you to state this relationship? (Multiple possible answers, pick one relationship to discuss.)

$X \sim Bin(n, p)$

$\frac{X}{n}$ is rescalled mean $p$

$\sigma = \sqrt{\frac{p(1-p)}{n}}$

1. By Weak LLN,

$\frac{X}{n} \xrightarrow{p} p.$

2. By CLT,

$Z = \frac{\frac{X}{n} - p}{\sqrt{\frac{p(1-p)}{n}}} \Rightarrow N(0,1)$

(convergence in distribution)

Note, we know we want $np, n(1-p) \geq 10$ for good approx.