Name: Solutions

Math 29 – Probability

Practice Final Exam

Instructions:

1. Show all work. You may receive partial credit for partially completed problems.
2. You may use calculators and a two-sided sheet of reference notes. You may not use any other references or any texts, except the provided z-table.
3. You may not discuss the exam with anyone but me.
4. Suggestion: Read all questions before beginning and complete the ones you know best first. Point values per problem are displayed below if that helps you allocate your time among problems.
5. You need to demonstrate that you can solve all integrals in problems that do not have a (DO NOT SOLVE) statement. I.E. write out some work showing how you solved the integration, including if necessary integration by parts.
6. Good luck!

Problem	1	2	3	4	5	6	7	8	9	Total
Points Earned										
Possible Points	10	11	10	10	18	10	10	4	6	90

Note: The points total on your final will be 100 points. This was my fast attempt to assign points here, so you can see what I thought the problems were worth before any adjustments.

Bowl	1	2	3
White	1	2	3
Red	4	3	2

1. Three bowls are labeled 1, 2, and 3, respectively. Bowl *i* contains *i* white and 5-*i* red balls. In an experiment, a bowl is randomly selected from the set of three bowls. Then, 3 balls are randomly selected without replacement from the contents of the selected bowl.

a. Given that bowl 1 was NOT selected, what is the probability of drawing exactly 2 red balls?

$$P(2 \text{ red} \mid \text{Not bowl } 1) = \frac{1}{2}\frac{\binom{3}{2}}{\binom{5}{3}} + \frac{1}{2}\frac{\binom{2}{2}}{\binom{5}{3}}$$

$$= \frac{1}{2} \cdot \frac{1}{10}(3+1) = \frac{1}{5}$$

b. What is the probability that exactly 2 red balls are drawn?

$$P(\text{exactly } 2 \text{ red}) = \frac{1}{3}\frac{1}{\binom{5}{3}}\left(\binom{4}{2}+\binom{3}{2}+\binom{2}{2}\right)$$

$$= \frac{1}{30}(6+3+1) = \frac{10}{30} = \frac{1}{3}$$

c. Given that exactly 2 red balls were drawn, what is the probability that bowl 3 was selected?

$$P(\text{Bowl } 3 \mid \text{exactly } 2 \text{ red}) = \frac{P(\text{Bowl } 3 \text{ and } 2 \text{ red})}{P(2 \text{ red})}$$

$$= \frac{P(2 \text{ red} \mid \text{Bowl } 3)\, P(\text{Bowl } 3)}{1/3} = \frac{P(2 \text{ red} \mid \text{Bowl } 3)(1/3)}{1/3}$$

$$= P(2 \text{ red} \mid \text{Bowl } 3) = \frac{\binom{2}{2}}{\binom{5}{3}} = \frac{1}{10}$$

2. It is known that blank CDs produced by a certain company will be defective with probability .05, independently of each other. The company sells the blank CDs in packages of 10 and offers a money-back guarantee that at most 1 of the blank CDs in the package will be defective. You may assume that the number of defectives in packages are independent of each other. $X = \#$ def. $X \sim Bin(10, .05)$

a. What is the probability that a package is returnable under the terms of the money-back guarantee?

$P(X>1) = 1 - P(X \le 1) = 1 - (P(X=0) + P(X=1)) = 1 - (.9138)$

$P(X=0) = \binom{10}{0}(.95)^{10} = .5987$ $\qquad = .0862$

$P(X=1) = \binom{10}{1}(.05)(.95)^9 = .3151$

b. If someone buys 4 packages, what is the probability that s/he will be able to return 2 of the packages under the money-back guarantee? $W = \#$ returnable in 4 packages $\quad W \sim Binomial(4, .0862)$

$P(W=2) = \binom{4}{2}(.0862)^2(1-.0862)^2$

$= 6(.0862)^2(.9138)^2 = .0372$

c. The cost, C, to the manufacturer is given by $C = Y^2 + 5Y + 1$, where Y is the number of returnable (under terms of the money-back guarantee) packages shipped. Find the expected cost associated with a shipment that contains 100 packages of CDs.

$Y \sim Binomial(100, .0862)$

$C = Y^2 + 5Y + 1 \Rightarrow E(C) = E(Y^2 + 5Y + 1) = E(Y^2) + 5E(Y) + 1$

$E(Y) = 8.62 \qquad V(Y) = 7.877 \qquad E(Y^2) = 82.1814$

$\Rightarrow E(C) = 82.1814 + 5(8.62) + 1 = 126.2814$

$X \sim Gamma(2,4)$

3. Consider a random variable X with pdf given by $f(x) = kxe^{-x/4}$, x>0, and 0, otherwise.

a. Find the value of k that makes this a valid pdf.

K must be $\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} \Rightarrow \dfrac{1}{\Gamma(2)4^2} = \dfrac{1}{16}$

b. Find the mgf for X. You may either identify the distribution and provide its associated mgf or derive the mgf directly.

mgf for a Gamma is $(1-\beta t)^{-\alpha}$

$\Rightarrow M_X(t) = (1-4t)^{-2}$

c. Find the mean and variance of X ***using the moment generating function*** you found in b.

$E(X) \Rightarrow M_X'(t) = -2(1-4t)^{-3}(-4) = 8(1-4t)^{-3}$

$M_X'(0) = 8$ This is $\alpha\beta = 2(4) = 8$

$M_X''(t) = 8(-3)(1-4t)^{-4}(-4) = 96(1-4t)^{-4}$

$M_X''(0) = 96 = E(X^2)$

$V(X) = E(X^2) - [E(X)]^2 = 96 - 8^2 = 96 - 64 = 32$

This is $\alpha\beta^2 = 32.$ ✓

4. A company needs a vast amount of iron ore for a project. Suppose $X_1, X_2, \ldots, X_{40}$ are a random sample of measurements on the proportion of impurities in iron ore samples from "Ores R Us" (a supplier company). The proportion of impurities in the population of all similar iron ore samples, X, has pdf $f(x) = 3x^2, 0 < x < 1$, and 0, otherwise.

$X \sim Beta(3,1)$

a. The company will refuse to buy the ore if $\bar{X}$ exceeds .8. Find the approximate probability that $\bar{X}$ exceeds .8 for a sample of size 40.

$E(X) = \frac{3}{4} \quad V(X) = \frac{3}{4^2(5)} = \frac{3}{80}$

By CLT, $\bar{X} \approx N(\mu, \sigma/\sqrt{n})$

$\Rightarrow \bar{X} \approx N\left(3/4, \sqrt{3/80}/\sqrt{40} = .0306\right)$

$P(\bar{X} > .8) = P\left(Z > \frac{.8 - .75}{.0306}\right) = P(Z > 1.6339)$

$\approx P(Z > 1.63) = .0516$

b. What numerical value does $\bar{X}_{40}$ converge in probability to? Justify your answer.

Since $E(X) = \frac{3}{4}$ and $V(X) = \frac{3}{80} < \infty$,

$\bar{X}_{40} \xrightarrow{P} \frac{3}{4}$ by WLLN.

5. Suppose that in 2 neighboring counties, road repairs per week are classified as major if the amount of road that requires repairing is 3 miles or longer. Assume the repair lengths in both counties range between 0 and 4 miles (continuous, not discrete). Let X denote length of repairs in one county and Y denote length of repairs in the neighboring county for a given week. Assume the joint pdf of X and Y is given by $f(x, y) = xy/64, 0 \le x, y \le 4$, and 0, otherwise.

a. Find the joint pdf for Z=X+Y and W=Y. $Z = X + Y \quad W = Y$

$Y = W = h_2(z, w) \qquad X = Z - Y = Z - W = h_1(z, w)$

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \qquad |J| = 1$$

Bounds

$0 \le x \le 4$

$0 \le y \le 4$

$\Rightarrow$

$0 \le z - w \le 4$

$w \le z \le 4 + w$

$0 \le w \le 4$

$$f_{Z,W} = f_{X,Y}(z-w, w)\,|1|$$
$$= f_{X,Y}(z-w, w)$$
$$= \frac{(z-w)w}{64} = \frac{zw - w^2}{64}$$

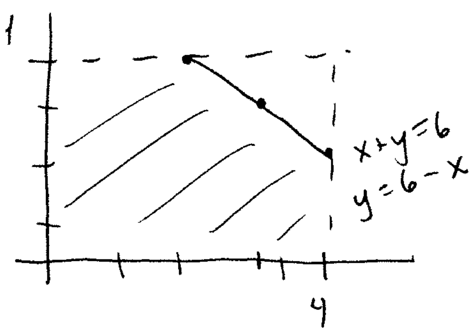
b. Show that X and Y are independent and have identical distributions (provide the marginal pdf they share).

$$f_X(x) = \int_0^4 \frac{xy}{64}\,dy = \frac{x}{64}\left(\frac{1}{2}y^2\right)\Big|_0^4 = \frac{x}{8}, \quad 0 \le x \le 4$$

$$f_Y(y) = \int_0^4 \frac{xy}{64}\,dx = \frac{y}{64}\left(\frac{1}{2}x^2\right)\Big|_0^4 = \frac{y}{8}, \quad 0 \le y \le 4$$

$$f_{XY} = \frac{xy}{64} = \frac{x}{8}\left(\frac{Y}{8}\right) = f_X f_Y \iff X \perp Y$$

c. The two counties want to hire a single company for the repairs. One particular company will only handle combined jobs of at most 6 miles at a time for a given week before charging huge additional fees. Using a probabilistic argument (i.e. compute a meaningful probability), would you recommend the counties use this company for their repairs?



$$P(X+Y \le 6) = \int_0^4 \int_0^{6-x} \frac{xy}{64}\,dy\,dx$$

$$= \int_0^4 \frac{x}{64}\left(\frac{1}{2}y^2\right)\Big|_0^{6-x} dx = \frac{1}{128}\int_0^4 x(6-x)^2\,dx$$

$$= \frac{1}{128}\int_0^4 x(36 - 12x + x^2)\,dx$$

$$= \frac{1}{128}\int_0^4 36x - 12x^2 + x^3\,dx = \frac{1}{128}\left(18x^2 - 4x^3 + \frac{1}{4}x^4\right)\Big|_0^4$$

$$= \frac{1}{128}\left(18(4^2) - 4^4 + \frac{1}{4}(4^4)\right) = \frac{1}{128}(96) = .75$$

6. Suppose X and Y are random variables where Var(X)=8 and Var(Y)=6.

a. If X and Y are independent, what is the variance of $6X+3Y+2$?

$$Var(6X+3Y+2) = 36\,Var(X) + 9\,Var(Y)$$
$$= 36(8) + 9(6) = 342$$

b. If X and Y have correlation .4, what is the variance of $X-2Y$?

$$Cov(X,Y) = .4(\sqrt{8})(\sqrt{6}) = 2.77$$

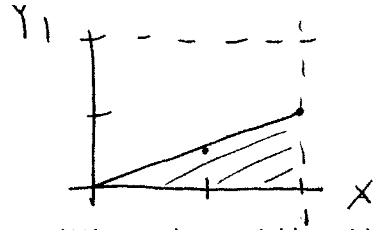
$$Var(X-2Y) = Var(X) + 4\,Var(Y) - 4\,Cov(X,Y)$$
$$= 8 + 4(6) - 4(2.77) = 20.92$$

c. (A Little Theory) If X and Y are independent random variables, show that $E(Y^3 \mid X) = E(Y^3)$. You may treat X and Y as continuous random variables, and use regular notation for their joint pdf and marginal pdfs.

$$X \perp Y \Leftrightarrow f_{XY} = f_X f_Y \Rightarrow f_{Y|X} = \frac{f_{X,Y}}{f_X} = \frac{f_X f_Y}{f_X} = f_Y$$

$$E(Y^3|X) = \int_Y y^3 f_{Y|X}\,dy = \int_Y y^3 f_Y\,dy = E(Y^3)$$

We know the bounds don't depend on X b/c $X \perp Y$.



$$y = \frac{1}{2}x$$
$$2y = x$$

7. Let X and Y be random variables with joint pdf $f(x,y) = x + y, 0 < x, y < 1$, and 0, otherwise.

a. Set up an integral (DO NOT SOLVE) to find the probability that X is greater than 2Y.

$$P(X > 2Y) = \int_0^1 \int_0^{\frac{1}{2}x} (x+y)\,dy\,dx \quad \text{OR} \quad \int_0^{\frac{1}{2}} \int_{2y}^{1} (x+y)\,dx\,dy$$

b. Find the conditional distribution of Y given X.

$$f_{Y|X} = \frac{f_{X,Y}}{f_X} = \frac{x+y}{x+\frac{1}{2}}, \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

$$f_X = \int_0^1 (x+y)\,dy = xy + \frac{1}{2}y^2 \Big|_0^1 = x + \frac{1}{2}, \quad 0<x<1$$

c. Find the probability that Y is greater than .25 given that X is .5.

$$f_{Y|X=.5} = \frac{y+\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} = y + \frac{1}{2}, \quad 0<y<1$$

$$P(Y > .25 \mid X = .5) = \int_{.25}^{1} \left(y + \frac{1}{2}\right) dy = \frac{1}{2}y^2 + \frac{1}{2}y \Big|_{.25}^{1}$$

$$= 1 - \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \frac{1}{4}\right) = 1 - \frac{1}{2}\left(\frac{5}{16}\right) = 1 - \frac{5}{32} = \frac{27}{32}$$

d. Find the expected value of Y given that X is .75.

$$f_{Y|X=.75} = \frac{y+\frac{3}{4}}{\frac{1}{2}+\frac{3}{4}} = \frac{4y+3}{2+3} = \frac{4y+3}{5}$$

$$E(Y \mid X = .75) = \int_0^1 y\left(\frac{4y+3}{5}\right) dy = \frac{1}{5}\int_0^1 4y^2 + 3y\,dy$$

$$= \frac{1}{5}\left(\frac{4}{3}y^3 + \frac{3}{2}y^2\right)\Big|_0^1 = \frac{1}{5}\left(\frac{4}{3}+\frac{3}{2}\right) = \frac{1}{5}\left(\frac{8}{6}+\frac{9}{6}\right) = \frac{17}{30}$$

8. The class is throwing a celebratory party for the end of the semester. A large number of pizzas are ordered – 40% from Antonio's and the rest from Domino's. Of the Domino's pizzas, 30% are cheese only while the rest have some toppings. From Antonio's, only 15% are cheese only. What is the probability that a pizza came from Antonio's if it is known to have toppings besides cheese?

$P(A) = .4$ $\quad P(D) = .6$ $\quad P(C \mid D) = .3$ $\quad P(NC \mid D) = .7$

$P(C \mid A) = .15$ $\quad P(NC \mid A) = .85$

$$P(A \mid NC) = \frac{P(A \text{ and } NC)}{P(NC)} = \frac{P(A)\,P(NC \mid A)}{P(A)\,P(NC \mid A) + P(D)\,P(NC \mid D)}$$

$$= \frac{.4(.85)}{.4(.85) + .6(.7)} = \frac{.34}{.34 + .42} = .4474$$

9. Matching. (Not all choices may be used.)

D. A stochastic process where the random variables are related by conditional probabilities

A. A distribution that may be used to approximate the Poisson

G. Result related to convergence in probability

C. Combinatorial method that is employed when order of objects in a subset does matter

F. Example distribution where the mean doesn't exist

E. Example distribution where the mean and variance are equal

A. Normal
B. Combination
C. Permutation
D. Markov Chain
E. Poisson
F. Cauchy
G. Weak LLN
H. Gamma
I. Sample Mean