Homework 10 Solution for Math 27 — Not Collected

Chapter 7

50. $X_1,\dots,X_n \perp \text{Uni}(0,\theta)$

a. $f(x)=\frac{1}{\theta}$ $\quad F(x)=\int_0^x \frac{1}{\theta}\,dt=\frac{x}{\theta}$

$$g_n(x) = n\left[F(x)\right]^{n-1} f(x)$$
$$= n\left[\frac{x}{\theta}\right]^{n-1}\left(\frac{1}{\theta}\right) = \frac{n x^{n-1}}{\theta^n}, \quad x\in(0,\theta)$$

b. $$E\left[X_{(n)}\right] = \int_0^\theta \frac{nx^n}{\theta^n}$$
$$= \frac{n}{\theta^n}\frac{1}{n+1}x^{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$$

c. $$E\left[X_{(n)}^2\right] = \frac{n\theta^2}{n+2}$$
$$\Rightarrow V(X_{(n)}) = \frac{n\theta^2}{n+2} - \left(\frac{n}{n+1}\theta\right)^2$$
$$= \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)$$

d. (skip) soln. = 0 (easy)

e. $R = X_{(n)} - X_{(1)}$

$$g_{1n} = \frac{n!}{(n-2)!}\left[\frac{x_n}{\theta} - \frac{x_1}{\theta}\right]^{n-2}\frac{1}{\theta^2}$$
$$= \frac{n(n-1)}{\theta^n}(x_n - x_1)^{n-2}$$

X-Forms $\quad S = X_{(1)}$ temp variable

$X_{(1)} = S \qquad X_{(n)} = R+S$

$0 < s-r < s < \theta - r$

$0 < r < \theta - s$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad |J| = 1$$

$$f_{RS} = g_{1n}(s, r+s)$$
$$= \frac{n(n-1)}{\theta^n}(r+s-s)^{n-2} = \frac{n(n-1)}{\theta^n} r^{n-2}$$

$$f_R = \int_0^{\theta-r} \frac{n(n-1)}{\theta^n} r^{n-2}\,ds = \frac{n(n-1)}{\theta^n} r^{n-2}(\theta - r)$$

f. $$E(R) = \int_0^\theta \frac{n(n-1)}{\theta^n} r^{n-1}(\theta - r)\,dr = \left(\frac{n-1}{n+1}\right)\theta$$

Chapter 8

2.
$$E(X_i) = \int_{-1}^1 \frac{3}{2}x^3\,dx = \frac{3}{8}x^4\Big|_{-1}^1 = 0$$
$$E(X_i^2) = \int_{-1}^1 \frac{3}{2}x^4\,dx = \underset{<\infty}{\frac{3}{5}} = \sigma^2 \text{ b/c } E(X)=0$$

By WLLN,
$$\bar{X}_n \xrightarrow{P} 0$$

4. By properties of the Poisson, $E(X_i) = \lambda$ and $V(X_i) = \lambda < \infty$

So by WLLN, $\bar{X}_n \xrightarrow{P} \lambda$

8. $$E(X) = \int_2^\infty \frac{2}{x}\,dx = 2\ln x\Big|_2^\infty$$

$\Rightarrow$ DNE $\quad V(X)$ cannot exist

WLLN does not apply

16. $\mu = 12 \quad \sigma = 9 \quad n = 100$

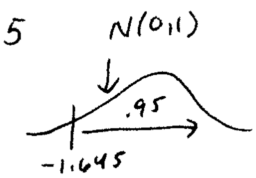
By CLT, $Y = \dfrac{\bar{X} - 12}{9/10} \sim N(0,1)$

$$P(\bar{X} > 14) = P\left(Y > \frac{14-12}{.9}\right) = P(Y > 2.22) = .0132$$

20. $\mu, \sigma^2 = 4 \quad n = 50$

$P(\text{Total} > 200) = .95$

$P(\bar{X} > 4) = .95$



$$-1.645 = \frac{4-\mu}{2/\sqrt{50}}$$

$$\mu = 4.4653$$

8.24

$$Y_1 = \frac{\bar{X}_1 - \mu_1}{\sigma_1/\sqrt{n_1}} \sim N(0,1)$$

$$Y_2 = \frac{\bar{X}_2 - \mu_2}{\sigma_2/\sqrt{n_2}} \sim N(0,1)$$

$\Rightarrow$

$\bar{X}_1 - \mu_1 \sim N(0, \sigma_1/\sqrt{n_1})$

$\bar{X}_2 - \mu_2 \sim N(0, \sigma_2/\sqrt{n_2})$

$\Rightarrow$

$$(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2) \sim N\left(0, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

$N(0, .3741657)$

So,

$P(|(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)| \leq .5)$

$P\left(|Z| \leq \frac{.5}{.3741657} = 1.336\right)$

$= .9082 - .0918 = .8164$

8.34 $X \sim Exp(10)$ $n = 100$

$P(\overset{Y=}{50}$ or more wait for > 10 min$)$

$P(1$ waits > 10 min$) = P(X > 10) =$

$\int_{10}^{\infty} \frac{1}{10} e^{-x/10} dx = .3679$ (rounded)

$Y \sim Bin(100, .3679)$ $P(Y \geq 50)$

check rule: $np = 36.79$, $n(1-p) \geq 10$ ✓

$\mu = 36.79$ $\sigma = \sqrt{np(1-p)} = 4.822$

$P(Y \geq 50) \overset{cc}{=} P(W \geq 49.5)$

$= P\left(Z \geq \frac{49.5 - 36.79}{4.822}\right) =$

$P(Z \geq 2.64) = .0041$

Use CC when $\approx$ a discrete dist with a continuous dist. You don't need it working with $\bar{X}$ b/c that's not discrete, but you do for counts/totals, etc.