Math Stat Handout

Suppose you are modeling battery lifetimes with an exponential distribution. So, $X_1, \dots, X_n$ is a random sample with $f(x) = \frac{1}{\beta} e^{-x/\beta}, x$ ($X_i \sim \text{Exp}(\beta)$).

a. Using all n obs., provide an unbiased pt. estimator for β. (Show it is unbiased)

b. Derive the MSE of your pt. estimator from a.

c. What is the distribution of $\sum_{i=1}^{n} X_i$?

d. Looking @ c, derive a pivotal quantity for use in estimating β with a CI. What is the distribution of your pivot?

e. Using either your answer to d., or a version of it, you should be able to get a χ^2 dist. for the pivot. Use this pivot and obtain a 95% CI for β when $\overline{X} = 23.45$ hrs for a RS of $n = 10$ batteries.

Math Stat Handout Solutions

$X_1, \dots, X_n \sim Exp(\beta)$

a. Try $\bar{X}$. $E(\bar{X}) = \frac{1}{n} E(\sum X_i) = \frac{1}{n} \sum E(X_i) = \frac{1}{n} \cdot n\beta = \beta$

$\bar{X}$ is unbiased for β.

b. $MSE(\bar{X}) = Var(\bar{X}) + (\cancel{B(\bar{X})})^{2}{}^{0} = Var(\bar{X}) = \frac{\beta^2}{n}$

c. By rules on adding Gammas (b/c $Exp(\beta) = Gamma(1, \beta)$)

$Y = \sum X_i \sim Gamma(n, \beta)$

d. If we eliminated dependence on β we could get $Gamma(n, \_\_)$

mgf for $Gamma(n, \beta) = (1-\beta t)^{-n} = M_Y(t)$

dividing by β would do... $U = \frac{Y}{\beta}$

$M_U(t) = (1 - \beta(\frac{1}{\beta}t))^{-n} \Rightarrow (1-t)^{-n} \sim Gamma(n, 1)$

So $\frac{Y}{\beta}$ is a pivot $\frac{\sum X_i}{\beta}$

e. To get a χ^2 dist, that's $Gamma(\frac{\nu}{2}, 2)$ so, we need to multiply by 2.

$V = \frac{2\sum X_i}{\beta}$ $\quad M_V(t) = (1 - \beta(\frac{2}{\beta}t))^{-n} = (1-2t)^{-n}$

$\sim Gamma(n, 2)$

$\sum X_i = n\bar{X} = 10(23.45) = 234.5$

$\Rightarrow \chi^2(2n)$

$P(a \le V \le b) = .95 \Rightarrow P(\chi^2_{.975}(20) \le \frac{2\sum X_i}{\beta} \le \chi^2_{.025}(20)) = .95$

$P(9.59083 \le \frac{469}{\beta} \le 34.1696) \Rightarrow P(\frac{1}{34.1696} \le \frac{\beta}{469} \le \frac{1}{9.59083})$

$\Rightarrow$ 95% CI is $\left(\frac{469}{34.1696}, \frac{469}{9.59083}\right) \Rightarrow (13.72565, 48.90088)$