Name: Solutions

Math 30 – Mathematical Statistics

Midterm 1 Practice Exam 2

Instructions:

1. Show all work. You may receive partial credit for partially completed problems.
2. You may use calculators and a one-sided sheet of reference notes. You may not use any other references or any texts, apart from the provided tables (Chi-square), and distribution sheet.
3. You may not discuss the exam with anyone but me.
4. Suggestion: Read all questions before beginning and complete the ones you know best first. Point values per problem are displayed below if that helps you allocate your time among problems, as well as shown after each part of the problem.
5. Good luck!

Problem	1	2	3	Total
Points Earned				
Possible Points	12	11	12	35

1. Suppose that $Y_1, Y_2, ..., Y_n$ are a random sample from an exponential distribution with parameter θ.

a. Show that $Y_{(1)} = \min(Y_1, Y_2, ..., Y_n)$ is biased for θ. (8)

$$f(y) = \frac{1}{\theta} e^{-y/\theta}$$

$$F(y) = \int_0^y \frac{1}{\theta} e^{-t/\theta} dt = 1 - e^{-y/\theta}$$

$$f_{Y_{(1)}} = n\left[1 - F(y)\right]^{n-1} f(y)$$

$$= n\left[e^{-y/\theta}\right]^{n-1} \frac{1}{\theta} e^{-y/\theta}, \ y > 0$$

$$= \frac{n}{\theta} e^{-\frac{ny}{\theta}}, \ y > 0$$

Side note: $\left(Y_{(1)} \text{ is } Exp\left(\frac{\theta}{n}\right)\right)$

$$E(Y_{(1)}) = \frac{n}{\theta}\int_0^\infty y e^{-\frac{ny}{\theta}} dy \qquad \begin{matrix} w = y \\ \alpha = 2 \\ \beta = n/\theta \end{matrix} \qquad \int_0^\infty w^{\alpha-1} e^{-\beta w} dw = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

$$= \frac{n}{\theta}\left(\frac{\Gamma(2)}{(n/\theta)^2}\right) = \frac{n}{\theta} \cdot \frac{\theta^2}{n} = \frac{\theta}{n} \neq \theta$$

$\Rightarrow Y_{(1)}$ is biased for θ.

b. Find a function of $Y_{(1)}$ that is unbiased for θ. Denote this new estimator $\hat{\theta}$. (2)

$\hat{\theta} = nY_{(1)}$ is unbiased for θ.

c. Derive the MSE of $\hat{\theta}$. (2)

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [B(\hat{\theta})]^2 = Var(\hat{\theta})$$

$$= Var(nY_{(1)}) = n^2 Var(Y_{(1)}) = n^2\left(\frac{\theta}{n}\right)^2 = \theta^2$$

2. Suppose Y is a random variable with a Gamma(6, β) distribution, where β is unknown.

a. Find a pivotal quantity (show it is a pivot) that has a chi-square distribution that could be used to find a confidence interval for β. (5)

$Y \sim Gamma(6, \beta)$ $\qquad M_Y(t) = (1-\beta t)^{-6}$

Let $Y^* = \frac{2Y}{\beta}$, $\qquad$ Need 2 instead of β for χ^2

$$M_{Y^*}(t) = \left(1 - \beta\left(\frac{2}{\beta}\right)t\right)^{-6} = (1-2t)^{-6} \sim Gamma(6,2)$$

$$\Rightarrow \chi^2(12)$$

b. Use your pivot from a. to find a 90% confidence interval for β, assuming you sampled Y=25.67. If you could not find a pivot in a., pretend you have one that has a β in its denominator and follows a chi-square distribution. (6) $\quad \chi^2_{.05}(12) = 21.0261 \quad \chi^2_{.95}(12) = 5.22603$

$$P\left(5.22603 \leq \frac{2Y}{\beta} \leq 21.0261\right) = .90$$

$$P\left(\frac{1}{21.0261} \leq \frac{\beta}{2Y} \leq \frac{1}{5.22603}\right) = .90$$

$$\left(\frac{2Y}{21.0261}, \frac{2Y}{5.22603}\right) \Rightarrow (2.441727, 9.8239)$$

[Side note: $Y = 25.67$ was generated from a Gamma$(6, 4)$ so $\beta = 4$ is contained in our CI which is good!]

3. Suppose you have a random sample of n observations from an exponentially distributed population with mean θ.

a. Find the method of moments estimator of θ^2. (3)

$\mu_2' =$

$E(Y) = \theta \qquad Var(Y) = \theta^2 = E(Y^2) - \theta^2 \Rightarrow E(Y^2) = 2\theta^2$

So, relate to 2^{nd} moment:

$m_2' = \frac{1}{n}\sum Y_i^2 = 2\theta^2 \qquad \widehat{(\theta^2)} = \frac{m_2'}{2} = \frac{1}{2n}\sum(Y_i^2)$

b. Find the MLE of θ^2. (5)

$f(y|\theta) = \frac{1}{\theta}e^{-y/\theta} \qquad L(\theta) = \frac{1}{\theta^n}e^{-\sum y_i/\theta}$

$l(\theta) = -n\ln\theta - \frac{\sum y_i}{\theta}$

$l'(\theta) = \frac{-n}{\theta} + \frac{\sum y_i}{\theta^2} \Rightarrow \frac{n}{\theta} = \frac{\sum y_i}{\theta^2} \qquad n\theta = \sum y_i$

$\hat{\theta} = \frac{\sum y_i}{n} = \bar{Y}$

MLE for θ^2 is $(\text{MLE for } \theta)^2$ by invariance property

So MLE of θ^2 is $\bar{Y}^2$

c. Is your MLE in b. minimal sufficient for θ^2? Explain why it is or explain why not. (2)

$\sum y_i$ is suff. b/c of FC for θ, so $\bar{Y}$ is also suff as 1-1 of $\sum y_i$. $\Rightarrow \bar{Y}^2$ is suff for θ^2 → use $\sqrt{\bar{Y}^2}$ etc.

(you could write $L(\theta) = g(\bar{Y}^2, \theta^2)\cdot h(y)$ for FC too)

If an MLE is suff, it is min suff.

d. Is your MLE in b. MVUE for θ^2? Show it is or explain why not. (2)

Need only to show unbiased for MVUE b/c it is suff.

$$E(\bar{Y}^2) = Var(\bar{Y}) + (E(\bar{Y}))^2 = \frac{\sigma^2}{n} + \mu^2 = \frac{\theta^2}{n} + \theta^2$$

$\Rightarrow$ NOT unbiased for $\theta^2 \Rightarrow$ not MVUE