Name: Solutions

Math 29 – Probability

**Practice** Third Midterm Exam 1

Instructions:

1. Show all work. You may receive partial credit for partially completed problems.
2. You may use calculators and a one-sided sheet of reference notes. You may not use any other references or any texts.
3. You may not discuss the exam with anyone but me.
4. Suggestion: Read all questions before beginning and complete the ones you know best first. Point values per problem are displayed below if that helps you allocate your time among problems.
5. You need to demonstrate that you can solve all integrals in problems that do not have a (DO NOT SOLVE) statement. I.E. write out some work showing how you solved the integration, including if necessary integration by parts.
6. Good luck!

| Problem | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points Earned | | | | | |
| Possible Points | 20 | 10 | 10 | 10 | 50 |

↑ something similar

1. Let X and Y be two random variables such that their joint pdf is given by:

$f_{X,Y}(x,y) = xe^{-x(y+1)}, x > 0, y > 0$ and 0, otherwise.

a. Find the distribution (i.e. the pdf) of Z=XY.

Either condition or 2D xform.
Conditioning turns out to NOT be so nice here, unless you condition on

2D X form

$W = Y$ as temp second variable

$Y = W = h_2(z,w)$ and $X = \frac{z}{w} = h_1(z,w) = zw^{-1}$

$$J = \begin{vmatrix} \frac{1}{w} & -\frac{z}{w^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{w} \qquad |J| = \frac{1}{w}$$

Bounds

$w > 0$ from $y > 0$

$\frac{z}{w} > 0 \Rightarrow z > 0$ from $x > 0$

So,

$$f_{Z,W} = f_{X,Y}\left(\frac{z}{w}, w\right) \cdot \frac{1}{w}$$

$$= \frac{1}{w}\left(\frac{z}{w}\right) e^{-\frac{z}{w}(w+1)} = \frac{z}{w^2} e^{-z-\frac{z}{w}}$$

Now need marginal of z

$$f_Z = \int_0^\infty \frac{z}{w^2} e^{-z-\frac{z}{w}} dw = e^{-z}\int_0^\infty \frac{z}{w^2} e^{-\frac{z}{w}} dw$$

u sub $u = -\frac{z}{w}$ $du = \frac{z}{w^2} dw$

$$= e^{-z}\int e^{u} du$$

$$= e^{-z} e^{u} = e^{-z} e^{-\frac{z}{w}}\Big|_0^\infty = e^{-z}(1-0) = e^{-z}, z$$

improper, technically
taking limits

So, $f_Z = e^{-z}, \quad z > 0$
$\quad 0, \quad$ o.w.

i.e. $Z \sim Exp(1)$

1. continued

b. Set up (DO NOT SOLVE) a double integral to find E(XY) using the joint pdf.

$$\int_0^\infty \int_0^\infty xy \left(xe^{-x(y+1)}\right) dx\, dy \quad \text{OR} \quad \int_0^\infty \int_0^\infty x^2 y e^{-x(y+1)} dx\, dy$$

can do x & y in any order.

c. Find E(XY) using Z in some fashion (Hint: DO NOT try to solve b., there is another way).

Note: $Z = XY$, so $E(XY) = E(Z)$.

$Z \sim \text{Exp}(1)$, so $Z$ has mean 1.

Hence $E(Z) = 1$.

d. Find the conditional pdf of Y given X.

$$f_{Y|X} = \frac{f_{X,Y}}{f_X} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}, \quad x>0,\ y>0$$

Find $f_X$.

$$f_X = \int_0^\infty xe^{-x(y+1)} dy = xe^{-x}\int_0^\infty e^{-xy} dy$$

$$= xe^{-x}\left(-\frac{1}{x}\right)e^{-xy}\Big|_0^\infty = -e^{-x}(0-1) = e^{-x}, \quad x>0$$

$0$, o.w.

$X$ is $\text{Exp}(1)$

2. a. Find the expected value of $X^3$ given Y when the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{e^{-y}}{y}, 0 < x < y < \infty \text{ and } 0\text{, otherwise.}$$

(For partial credit, you may state the conditional pdf of X given Y.)

$$f_{X|Y} = \frac{f_{X,Y}}{f_Y} = \frac{\frac{e^{-y}}{y}}{e^{-y}} = \frac{1}{y}, \quad 0 < x < y \qquad X|Y \sim Uni(0,y)$$

$$f_Y = \int_0^y \frac{e^{-y}}{y}\,dx = \frac{x}{y} e^{-y}\Big|_0^y = e^{-y} \qquad Y \sim Exp(1)$$

$$E(X^3|Y) = \int_0^y x^3 \cdot \frac{1}{y}\,dx = \frac{1}{y}\frac{x^4}{4}\Big|_0^y = \frac{y^3}{4}$$

b. Set up (DO NOT SOLVE) an integral to find the expected value of $X^2 + e^X + 12$ given Y.

$$\int_0^y (x^2 + e^x + 12)\,\frac{1}{y}\,dx$$

3. Resistors in a circuit have an average resistance of 190 ohms ($\mu$) and a standard deviation of 8 ohms ($\sigma$). Suppose 40 resistors are randomly selected for use. Note that resistance is a continuous random variable.

$n = 40$

$\Rightarrow$ No Continuity correction

a. What is the probability that the average resistance for the set of resistors is between 188 and 191 ohms?

By CLT,

$\bar{X}$ is $\approx N(190, 8/\sqrt{40})$

$\Rightarrow N(190, 1.265)$

where $\dfrac{\bar{X} - 190}{1.265} = Z \sim N(0,1)$

$$P(188 \le \bar{X} \le 191)$$

$$= P\left(\frac{188-190}{1.265} \le Z \le \frac{191-190}{1.265}\right)$$

$$= P(-1.58 \le Z \le .79)$$

$$= P(Z \le .79) - P(Z \le -1.58) = .7852 - .0571 = .7281$$

b. What is the probability that the total resistance for the set of resistors does not exceed 7630 ohms?

$P(\text{Total} < 7630) \Rightarrow$ write in terms of an average

$$P(\bar{X} < 190.75) = P\left(Z < \frac{190.75 - 190}{1.265}\right)$$

$$= P(Z < .59) = .7224$$

4. Let $X_1, X_2, \ldots, X_5$ be independent Geo(.6) random variables. Prove (i.e. demonstrate with some work) that Y= $X_1 + X_2 + X_3 + X_4 + X_5$ is Negative Binomial (r=5, p=.6) with some justification for your steps.

Use method of Mgfs. We know if $X_1, X_2, \ldots, X_5$ are $\perp$ and $Y = \sum_{i=1}^{5} X_i$ then $M_Y(t) = \prod_{i=1}^{5} M_{X_i}(t)$.

So, since $M_{X_i}(t) = \dfrac{pe^t}{1-(1-p)e^t}$, $p = .6$ $\forall$ $i = 1, \ldots, 5$,

$$M_Y(t) = \prod_{i=1}^{5} \frac{.6e^t}{1-.4e^t} = \left(\frac{.6e^t}{1-.4e^t}\right)^5$$

This resembles the mgf for a Neg Bin $(5, .6)$ RV.

Since mgfs are unique, we conclude $Y$ is $NB(r=5, p=.6$