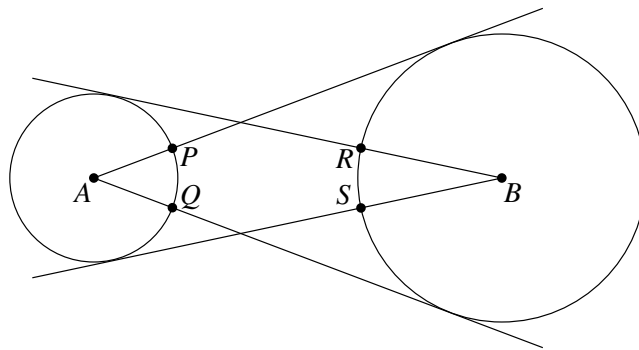


Walker Prize Examination

1. In Pascal's triangle, are there any spots where three consecutive entries in the same row are in the proportions 1:2:3? Equivalently, is there a choice of n and k so that the coefficients of x^k , x^{k+1} , and x^{k+2} in the expansion of $(1+x)^n$ are in the ratio 1:2:3? If so, give an example; if not, prove it never happens.
2. If 5 X's and 4 O's are placed at random in a standard three-by-three tic-tac-toe board (i.e., one symbol in each block of a 3×3 grid), what is the probability that some row, column, or diagonal will consist of three O's?
3. Alice and Bob sold their teddy bear collection, and by coincidence, the number of dollars they sold each bear for was the same as the number of bears. To split the money from the sale between them, they alternated taking \$10 each — first Alice took \$10, then Bob took \$10, then Alice took another \$10, and so on. At Bob's last turn, the amount of money left was less than \$10, so he simply took what was left. Alice, having taken more total money, promptly gave Bob a little of her money. After that transaction, they each had exactly half the money from the sale. How much money did Alice give Bob at the end?
4. A and B are centers of circles C_A and C_B which do not intersect each other, and neither of which sits inside the other. The two lines through A tangent to C_B intersect C_A at P and Q , while the two lines through B tangent to C_A intersect C_B at R and S , as shown in the picture. Show that the distance from P to Q is the same as the distance from R to S .



5. How many different real numbers x are there such that $100 \sin x = x$?
6. A regular tetrahedron is a solid with four faces, each of which is an equilateral triangle. If a regular tetrahedron is inscribed in a sphere of radius 1, meaning that each of the four vertices of the tetrahedron is a point on the surface of the sphere, what is the volume of the tetrahedron?

7. Let f be a function from \mathbb{R} to \mathbb{R} with the following two properties:

- (a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and
- (b) $[f(x)]^2 = f(x^2)$ for all $x \in \mathbb{R}$.

Prove that $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

8. For any sequence $\{a_n\}_{n \geq 1}$ of **nonnegative** real numbers, consider the two series

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} a_n^2.$$

- (a) Find $a_1, a_2, \dots \geq 0$ such that the first series converges but the second series diverges.
- (b) Find $a_1, a_2, \dots \geq 0$ such that the first series diverges but the second series converges.

9. Let n be a positive integer, and let $S = \{1, 2, \dots, 2n\}$. Let T be any subset of S containing at least $n + 1$ members of S . Prove that there are (at least) two different numbers $a, b \in T$ such that a divides b .

10. Let $f(x) = \int_1^x \frac{\ln t}{1+t} dt$. Compute $f(e) + f\left(\frac{1}{e}\right)$.