AMHERST COLLEGE

DEPARTMENT OF MATHEMATICS

WALKER PRIZE EXAMINATION

for

FIRST YEAR STUDENTS AND SOPHOMORES

On

Monday, April 16, 2012

7:00 – 10:00 pm

206 Seeley Mudd Building

- No books, notes or calculators are to be used.
- All ten questions have the same value, and you are not expected to answer them all.
- Make a careful selection of questions you find interesting and/or accessible.
- Answer the questions in any order and by any method.
- In grading, great importance will be attached to clear presentation of your argument.
- For scratch work you may use a spare blue book, or scratch paper.
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Walker Prize Examination – April 16, 2012

1. How many integers from 1 to 1,000,000 (one million) have the digit 3 but *not* the digit 5 appear when written in the usual base ten way?

2. For what values of the real number u_0 does the sequence defined recursively by the formula $u_{n+1} = u_n^2 - u_n$ converge to 0?

3. 100 people are lined up waiting to board a plane. The seats on the plane are numbered from 1 to 100, and each person's ticket says which seat he or she is supposed to sit in. (The people are *not* necessarily lined up by ticket number.) The first person in line ignores his ticket and chooses a seat at random. After that, each person sits in his or her assigned seat if it is available; if it is already occupied, then that person chooses an empty seat at random. What is the probability that the 100th person sits in her assigned seat?

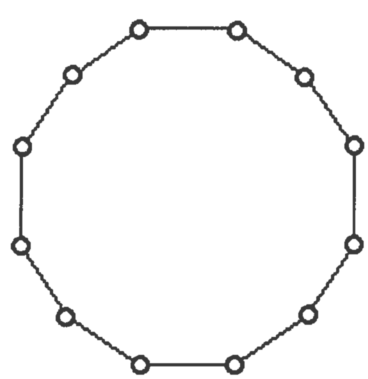
4. You're standing next to a pyramid. The base of the pyramid is a square, 80 ft on a side. The distance from each corner of the base to the point at the top of the pyramid is 70 ft. You want to start at some point along the base of the pyramid and walk up one of the faces in a straight line to the top of the pyramid, and you want your path to make a 45 degree angle with the ground. How far from a corner of the base should you start?

5. Prove that every polynomial can be written as a difference of two polynomials, each of which is an increasing function.

6. Alice and Bob are going to play the following game. They will take turns coloring in vertices, one at at time, of the dodecagon (12-side polygon) below. If the final two uncolored vertices are next to each other, then Alice wins; otherwise, Bob wins.

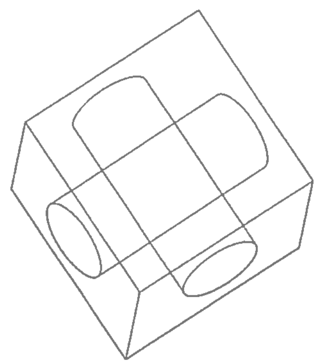
   A. Devise a winning strategy for Alice if she goes second.

   B. Devise a winning strategy for Bob if he goes second.



7. You need to weigh four hippos. The only scale available is a truck scale that starts at 350 kg., which is more than any of the hippos weigh. You decide to weigh them in pairs. You get the weights (in kg) 362, 516, 406, 428, and 494, but then when you try to weigh the last pair, which is the heaviest pair, the scale breaks. What are the weights of the hippos?

8. A solid wooden cube has side length 4 inches. Two cylindrical tunnels, each of radius 1 inch, are drilled through the cube. Each tunnel runs from the center of one face to the center of the face directly opposite; see the figure below. What is the volume of the solid that remains after the two tunnels have been drilled out of the cube?



9. Let x be a real number for which $\sin(x) + \cos(x) = 4/3$. Find the exact value of $\sin^3(x) + \cos^3(x)$.

10. Compute the value of $\int_0^{\pi/2} \ln \cos x \, dx$.

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Walker Prize Examination – April 21, 2011

1. Two ferry boats are based on opposite sides of a lake, one on the west side and one on the east. They start running at 8:00 a.m. each morning. Each boat goes to the base on the opposite side, spends an hour unloading and loading, and then returns to its home base. Each boat goes at a constant speed throughout its trip, but the speeds of the two boats may be different. On the first trip, when the boats pass each other, they are 8 miles from the eastern shore. When they are both on their return trip they pass again, this time 4 miles from the western shore. What is the width of the lake?

2. A cake has been cut into nine pieces, possibly weighing different (positive) amounts. Show that it is possible to cut one of the pieces into two pieces (with positive, but possibly different, weights) in such a way that you can then divide the resulting ten pieces of cake into two piles, with each pile containing five pieces and the two piles having the same total weight.

3. Evaluate $\int_0^1 \sin\left(\frac{\pi}{2}x^3\right) + \sqrt[3]{\frac{2}{\pi}\sin^{-1} x}\, dx$.

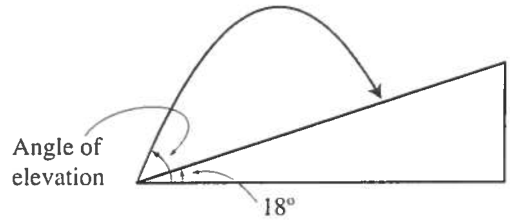
4. For any integer $n \geq 1$, let

$$a_n = \begin{cases} \dfrac{1}{(n+2)2^n}, & \text{if } n \text{ is odd,} \\ \dfrac{n-1}{(n+1)2^n}, & \text{if } n \text{ is even.} \end{cases}$$

Compute the sum of the series $\sum_{n=1}^{\infty} a_n$.

5. Assume the earth is a sphere of radius $r = 6000$ km. You are in an airplane 5 km above the surface of the earth. What fraction of the earth's surface can you see?

6. The stands at a stadium slope upward at an angle of 18°. You are firing a T-shirt cannon from the bottom of the stands into the stands. At what angle of elevation should you fire the cannon to get the T-shirt to land as far up the stands as possible? You may ignore air resistance.



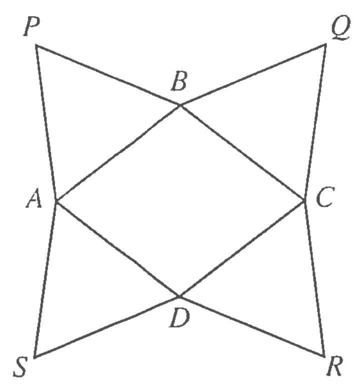
7. A company has 5 officers: a president, and 1st, 2nd, 3rd, and 4th vice-presidents. They have \$1000 to be distributed to the officers as bonuses; each officer will get a nonnegative integer number of dollars, and the total of the bonuses must be \$1000. They use the following procedure to determine the bonuses: The president proposes a distribution of the bonus money, and all of the officers vote on this proposal. If at least half of the officers vote yes, then the proposal is approved. If not, then the president is fired, and all other officers move up one rank: the 1st vice-president becomes president, the 2nd vice-president becomes 1st vice-president, and so on. Then the new president makes a proposal, and the (remaining) officers vote on it. Again, if at least half vote yes then the proposal is approved, and if not then the president is fired and all officers move up again. This procedure is repeated until a proposal is approved. Each officer can always be counted on to vote in his own best interest, which is defined as follows:

   - He doesn't want to be fired.
   - As long as he's not fired, he wants as much money as possible.
   - As long as he's getting as much money as possible, he'd also like to be promoted to the highest possible position.

   Each officer can also be counted on to figure out the best strategy for the sequence of votes, taking into account the voting strategies of the other officers. How will the president propose to distribute the bonuses?

8. In a round-robin tournament with n teams, each team plays every other team exactly once. There are no ties. Prove that either there are three teams A, B, and C such that A beats B, B beats C, and C beats A, or it is possible to list the teams in some order $T_1, T_2, \ldots, T_n$ such that for all i and j between 1 and n, T_i beats T_j if and only if $i < j$.

9. All line segments in the figure below have length 13. Given that the distance from P to Q is 24, compute the area of $ABCD$.



10. Find all continuous, one-to-one functions f from the real numbers to the real numbers such that the range of f is the entire number line $(-\infty, \infty)$ and for every number x,

$$f(x) = \frac{f^{-1}(x) - f^{-1}(-x)}{2}.$$

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Tuesday, April 20, 2010

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Walker Prize Examination – April 20, 2010

1. Determine whether the following series converges or diverges:

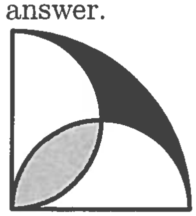
$$\sum_{n=1}^{\infty}\left(e^{1/n^2}-1\right).$$

Justify your answer.

2. For any real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Prove that for every real number x,

$$\lfloor x \rfloor + \left\lfloor x+\frac{1}{2010}\right\rfloor + \left\lfloor x+\frac{2}{2010}\right\rfloor + \cdots + \left\lfloor x+\frac{2009}{2010}\right\rfloor = \lfloor 2010x \rfloor.$$

3. The figure below shows a quarter of a circle of radius 1, and two half-circles, each of radius 1/2. Is the area with dark shading larger, smaller, or the same size as the area with light shading? Justify your answer.



4. A large (but finite) number of soldiers are arranged in an east-west line, and all the soldiers are facing north. The commander shouts "Right face!" One second later, all the soldiers ought to be facing east, but they have not completely mastered "right" and "left," so some are facing east and some west. Any soldier who is face-to-face with his neighbor realizes that there was a mistake and turns 180° (disregarding the possibility that the mistake might have been the neighbor's). One second later, when all these 180° turns have been completed, any soldier who is now face-to-face with a neighbor turns 180° (even if he had just turned at the previous step). The process repeats in the same manner. Prove that it stops after finitely many steps.

5. Prove that a positive integer n is prime if and only if there is a unique pair of positive integers j and k such that

$$\frac{1}{j}-\frac{1}{k}=\frac{1}{n}.$$

6. Let $\mathbb{Z}^+$ be the set of positive integers, and let f be a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$. Suppose that for all positive integers m and n,

$$f(m+f(n)) = f(m)+n.$$

Prove that for all positive integers n, $f(n)=n$.

7. A balance scale like the one in the picture below can be used, together with some known weights, to weigh objects. For example, if you have weights that weigh 2, 4, and 11 pounds, then you can determine whether or not an object weights 5 pounds by putting the object, together with the 2 and 4 pound weights, on one pan of the scale, and the 11 pound weight on the other pan. If the pans balance, then you know the object weighs exactly 5 pounds. Similarly, you can determine whether an object weighs 6 pounds by putting the object in one pan, and the 2 and 4 pound weights in the other. However, using these weights you cannot determine whether an object weighs exactly 1 pound.

   (a) Show that there is a set of 3 weights that can be used to determine the weight of any object whose weight, in pounds, is an integer from 1 to 13, inclusive.

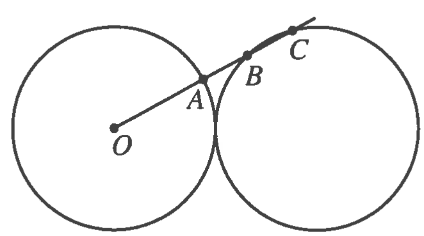
   (b) Show that there is no set of weights that can be used to determine the weight of any object whose weight, in pounds, is an integer from 1 to 14, inclusive.

8. Find the smallest possible value of the expression

$$\sqrt{1+x_0^2}+\sqrt{1+(x_1-x_0)^2}+\sqrt{1+(x_2-x_1)^2}+\cdots+\sqrt{1+(x_{10}-x_9)^2}+\sqrt{1+(x_{10}-5)^2},$$

where x_0, x_1, …, x_{10} are real numbers. Justify your answer.

9. Two circles of radius 1 are tangent, as shown in the figure. A ray from O, the center of the first circle, intersects the first circle at the point A and the second circle at the points B and C. If the lengths of the segments AB and BC are equal, find this common length.



10. Evaluate

$$\int_{1/2}^{2} \frac{\ln x}{x^2+1}\,dx.$$

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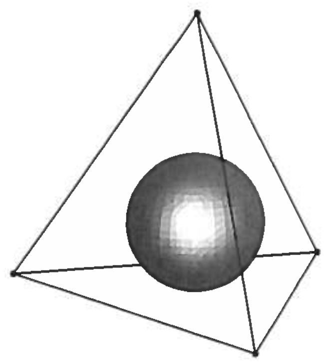
Walker Prize Examination – April 21, 2009

1. A positive integer is called *palindromic* if, when written in base 10, it reads the same forwards and backwards. For example, 1, 353, and 2772 are all palindromic. Find the 2009th palindromic positive integer.

2. A regular tetrahedron is a solid with four faces, each of which is an equilateral triangle. Consider a regular tetrahedron in which each edge has length 1. A sphere is inscribed in this tetrahedron so that it is tangent to all four faces. What is the radius of the sphere?



3. An ant crawls around the circle $x^2 + y^2 = 1$, and simultaneously another ant crawls along a curve C. When the first ant is at the point (x, y), the second ant is at the point $(x^4 - x^2y^2, 2x^3y)$. Show that C is a cardioid.

4. Alice says to Bob, "Let's play a game. We'll take turns rolling a die, and whoever rolls a 1 first wins. I'll roll first." (Recall that a die is a cube with the faces numbered from 1 to 6.) "But that's not fair," Bob replies. "If you go first, then you have a better chance of winning. Here's a better idea: We'll pick two positive integers a and b that are less than 6. You can go first, but whenever you roll the die, you have to get a or lower to win, and when I roll I have to get b or lower." Find integers a and b to ensure that Alice and Bob each have the same probability of winning this game.

5. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for every real number x and every integer m, $f(mx) = m^2 f(x)$. Be sure to give a proof that you have found *all* such functions.

6. Ten distinct positive integers are arranged in a row in such a way that the sum of any three consecutive numbers in the row is at least 20. Find the smallest possible sum of all ten numbers, and prove that no smaller sum is possible.

7. Find $\lim_{n\to\infty} n \sum_{k=n}^{2n} \frac{1}{k^2}$.

8. Some people sit in a circle. Each person has a pile of pennies, and the total number of pennies in all the piles is 128. A hat is put on one person's head. The person wearing the hat is called the payer, and if he has at least 64 pennies, then he must pay to each other person as many pennies as the other person already has. If the payer has fewer than 64 pennies, then no payment is made. After he has made these payments, the payer's turn is over (even if he still has at least 64 pennies) and he passes the hat to the person on his right. That person becomes the payer, and the process is repeated.

   For example, suppose there are 3 people: Alice has 44 pennies, Bob has 71, and Cathy has 13. Alice starts out as payer, but since she has fewer than 64 pennies, she makes no payment and passes the hat to Bob. Bob pays 44 pennies to Alice and 13 pennies to Cathy, so after the payment Alice has 88, Bob has 14, and Cathy has 26. Next Cathy becomes payer and makes no payment. Then Alice becomes payer again and pays 14 to Bob and 26 to Cathy. Now Alice has 48, Bob has 28, and Cathy has 52. Since everyone now has fewer than 64 pennies, no further payments will be made. In this example there were two turns in which money changed hands.

   (a) Prove that there can never be more than seven turns in which money changes hands.

   (b) Give an example of a starting configuration that leads to seven turns in which money changes hands.

9. Assume that there are 100 professors on the Amherst faculty. One day, the president announces that he wants to form a committee consisting of 34 professors. The next day, he receives 100 envelopes in the mail. He opens the first, and finds that it is a note from Professor Jones, saying "I refuse to be on the committee if Professor Smith is on it." When he opens the other envelopes, he finds that every professor has named exactly one other professor and said that he or she is unwilling to serve on the committee if that other professor is on it. Prove that it is possible for the president to form his committee without violating the wishes of any professor.

10. Find all real numbers $b > 0$ such that for all $x > 0$, $x^b \leq b^x$.