An Example of Thesis Formatting with LaTeX

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Abstract

This short example thesis meets all of the formatting requirements in Mathematics and shows how to use the blackboard bold font $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{N}$. Other fonts like calligraphic $\mathcal{K}$ and fraktur $\mathfrak{m}$ also appear, as do user-defined math operators like $\operatorname{diam} X$ and $\operatorname{div} \vec{F}$. We also model a writing style appropriate to a math thesis, although a real thesis should provide more exposition and justification than appears in, say, Chapter 3, which was taken largely from a published math research paper. Throughout, the reader should look at both the finished document and the raw LaTeX file to learn how to use LaTeX effectively. We recommend that the aspiring math thesis writer also consult the document `latextips.tex` for more examples of typeset mathematical equations, sectioning commands, and related techniques.

Acknowledgements

Thanks to Mom and Dad for everything. Thanks to Lyle McGeoch, who wrote the original 1990 sample `thesis.tex` file for CS theses on which this sample math thesis was based.

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List of Notation

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$\mathbb{F}_p$ . . . . . p. 6

$|\cdot|_v$ . . . . . p. 3

$|\cdot|_p$ . . . . . p. 4

K_v . . . . . p. 4

$\bar{K}_v$ . . . . . p. 4

$\mathbb{C}_v$ . . . . . p. 4

$\mathbb{Q}_p$ . . . . . p. 4

$\bar{\mathbb{Q}}_p$ . . . . . p. 4

$\mathbb{C}_p$ . . . . . p. 4

$D(a,r)$ . . . . . p. 4

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$\operatorname{diam} X$ . . . . . p. 5

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$\mathcal{K}_v$ . . . . . p. 6

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$\log_d x$ . . . . . p. 8

$E(N,d)$ . . . . . p. 8

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Chapter 1

Introduction

Let $\phi \in \mathbb{Q}(z)$ be a rational function with rational coefficients. Let ϕ^n denote the n^{th} iterate of ϕ under composition; that is, ϕ^0 is the identity function, and for $n \geq 1$, $\phi^n = \phi \circ \phi^{n-1}$. We will study the dynamics ϕ on the projective line $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. In particular, we say a point x is *preperiodic* under ϕ if there are integers $n > m \geq 0$ such that $\phi^m(x) = \phi^n(x)$. The point $y = \phi^m(x)$ satisfies $\phi^{n-m}(y) = y$ and is said to be *periodic* (of period $n - m$). Note that $x \in \mathbb{P}^1(\mathbb{Q})$ is preperiodic if and only if its orbit $\{\phi^n(x) : n \geq 0\}$ is finite.

Although ϕ can have infinitely many preperiodic points in $\mathbb{C}$, the same is not true over $\mathbb{Q}$, as the following result [3] states.

Theorem 1.0.1 (Northcott, 1950)**.** *Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 2$. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^1(\mathbb{Q})$.*

For example, let $\phi(z) = z^2 - 29/16$. Then the set $\{5/4, -1/4, -7/4\}$ forms a periodic cycle (of period 3), and $-5/4$, $1/4$, $7/4$, and $\pm 3/4$ each land on this cycle after one or two iterations. In addition, the point ∞ is of course fixed. These nine rational points are all preperiodic under ϕ. In fact, ϕ has only these nine rational preperiodic points [4, Theorem 3].

Morton and Silverman proposed the following Conjecture [2].

Uniform Boundedness Conjecture. (Morton and Silverman, 1994)
Given an integer $d \geq 2$, there is a constant $\kappa = \kappa(d)$ such that no rational function $\phi \in \mathbb{Q}(z)$ of degree at least d has more than κ preperiodic points in $\mathbb{P}^1(\mathbb{Q})$.

In this thesis we will study the Uniform Boundedness Conjecture by analyzing filled Julia sets $\mathcal{K}_v$, to be introduced in Definition 3.1.2. In Chapter 2 we will set terminology and discuss the geometry of subsets of certain fields. In Chapter 3 we will discuss some topics related to dynamics and prove our main results.

Chapter 2

Fundamentals

2.1 Absolute Values

Definition 2.1.1. Let K be a field. An *absolute value* v on K is a real-valued function $|\cdot|_v : K \to [0,\infty)$ such that for any $x, y \in K$, $|x| = 0$ if and only if $x = 0$, $|xy|_v = |x|_v|y|_v$, and $|x+y|_v \leq |x|_v + |y|_v$. If $|\cdot|_v$ satisfies the stronger triangle inequality $|x+y|_v \leq \max\{|x|_v, |y|_v\}$, then we say v is *non-archimedean*; otherwise we say v is *archimedean*.

Proposition 2.1.2. *Let K be a field with absolute value v, and let $x, y \in K$. Then:*

a. $|1|_v = 1$,

b. $|-x|_v = |x|_v$,

c. $|x - y|_v \leq |x|_v + |y|_v$,

d. *If v is* non-archimedean *and $|x|_v < |y|_v$, then $|x + y|_v = |x - y|_v = |y|_v$.*

Proof. To prove part (a), note that $|1|_v = |1 \cdot 1|_v = |1|_v|1|_v = |1|_v^2$, and hence either $|1|_v = 0$ or $|1|_v = 1$. However, since $1 \neq 0$ in the field K, we also have $|1|_v \neq 0$, according to Definition 2.1.1. Thus, $|1|_v = 1$, as desired.

For part (b), we begin by noting that $|-1|_v^2 = |(-1)^2|_v = |1|_v = 1$, and therefore either $|-1|_v = -1$ or $|-1|_v = 1$. However, $|-1|_v \in [0,\infty)$, and hence $|-1|_v = 1$. Thus, $|-x|_v = |-1|_v |x|_v = |x|_v$. Part (c) is now immediate, since $|x-y|_v \leq |x|_v + |-y|_v = |x|_v + |y|_v$.

Finally, to prove part (d), we have

$$|y|_v = |(x+y) - x|_v \leq \max\{|x+y|_v, |-x|_v\} = \max\{|x+y|_v, |x|_v\} \leq \max\{|x|_v, |y|_v\} = |y|_v,$$

and therefore $\max\{|x+y|_v, |x|_v\} = |y|_v$. However, $|x|_v < |y|_v$, implying that $|x+y|_v = |y|_v$. The proof that $|x-y|_v = |y|_v$ is similar. $\square$

The usual absolute value on $\mathbb{Q}$ is of course archimedean; we will denote it by $v = \infty$ or $|\cdot|_\infty$. Meanwhile, for any prime number p, the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ is given by $|0|_p = 0$ and $|p^e \cdot (r/s)|_p = p^{-e}$ for any integers e, r, s for which p does not divide r or s; it is a non-archimedean absolute value. See [1, 5] for more on p-adic absolute values.

Given a field K with an absolute value v, one can form the completion K_v of K with respect to v as the set of Cauchy sequences up to equivalence; K_v is a field, and v extends to it in a natural way [5, Proposition I.3.2]. In addition, v extends uniquely to an algebraic closure $\bar{K}_v$ of K_v [5, Proposition II.3.3]. However, $\bar{K}_v$ need not be complete.

Definition 2.1.3. Let K be a field with an absolute value v, and let $\bar{K}_v$ be an algebraic closure of K_v. We define $\mathbb{C}_v$ to be the completion of $\bar{K}_v$ with respect to v.

Fortunately, $\mathbb{C}_v$ is both complete and algebraically closed [5, Theorem III.3.3]. If $K = \mathbb{Q}$ and $v = \infty$ is the usual absolute value, then $K_v = \mathbb{R}$ and $\bar{K}_v = \mathbb{C}$ is already complete, and hence $\mathbb{C}_v = \mathbb{C}$. On the other hand, if $K = \mathbb{Q}$ and $v = p$ is the p-adic absolute value, then $K_v = \mathbb{Q}_p$ is the field of p-adic rational numbers, and the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is *not* complete [5, Corollary III.1.4]. Thus, its completion $\mathbb{C}_v = \mathbb{C}_p$ is a strictly larger field.

2.2 Radii and Diameters in $\mathbb{C}_v$

In this section, we fix a field $\mathbb{C}_v$ as in Definition 2.1.3. Given a point $a \in \mathbb{C}_v$ and a real number $r > 0$, we define the open and closed disks of radius r centered at a to be

$$D(a,r) := \{x \in \mathbb{C}_v : |x-a|_v < r\} \quad \text{and} \quad \overline{D}(a,r) := \{x \in \mathbb{C}_v : |x-a|_v \leq r\},$$

respectively.

Remark 2.2.1. If v is non-archimedean, the closed unit disk $\overline{D}(0,1)$ is often denoted $\mathfrak{o}_v$. It is a commutative ring with unity, since it is a subring of K; that is, it contains 1, is closed under multiplication, and, thanks to the non-archimedean property, is closed under addition and subtraction.

Meanwhile, the open unit disk $D(0,1)$ is often denoted $\mathfrak{m}_v$, and it is a maximal ideal of $\mathfrak{o}_v$. To see this, first note that $\mathfrak{m}_v$ is nonempty and closed under addition and subtraction, again by the non-archimedean property. Second, given any $a \in \mathfrak{o}_v$ and $b \in \mathfrak{m}_v$, we clearly have $|ab|_v = |a|_v|b|_v < 1$, and therefore $ba = ab \in \mathfrak{m}_v$, proving that $\mathfrak{m}_v$ is an ideal. Finally, if $I \supsetneq \mathfrak{m}_v$ is an ideal of $\mathfrak{o}_v$ properly containing $\mathfrak{m}_v$, then there is some $c \in I$ with $|c|_v = 1$. It follows that $|c^{-1}|_v = 1$, and hence $c^{-1} \in \mathfrak{o}_v$. Therefore, for any $a \in \mathfrak{o}_v$, because $ac^{-1} \in \mathfrak{o}_v$, we have $a = (ac^{-1})c \in I$. Thus, $I = \mathfrak{o}_v$, proving that $\mathfrak{m}_v$ is in fact a maximal ideal.

Just as in $\mathbb{R}$ or $\mathbb{C}$, the *diameter* of a set $X \subseteq \mathbb{C}_v$ is defined to be

$$\operatorname{diam} X := \sup\{|x-y|_v : x, y \in X\}$$

if $X \neq \varnothing$, or 0 if $X = \varnothing$. If X is bounded, i.e., if there is some $M > 0$ such that $|x|_v \leq M$ for all $x \in X$, then clearly X has finite diameter, since $|x-y|_v \leq |x|_v + |y|_v \leq 2M$. However, we will actually be interested in a different way to assign a size to X, as follows.

Definition 2.2.2. Let $X \subseteq \mathbb{C}_v$. The *radius* of X, denoted $\operatorname{rad} X$, is the the infimum of the radii of all closed disks containing X. That is,

$$\operatorname{rad} X := \inf\{r > 0 : X \subseteq \overline{D}(a,r) \text{ for some } a \in \mathbb{C}_v\}.$$

Note that if X is not a bounded set, then $\operatorname{rad} X = \inf \varnothing = \infty$.

Proposition 2.2.3. *Let $X \subseteq \mathbb{C}_v$. Then*

$$\operatorname{rad} X \leq \operatorname{diam} X \leq 2 \operatorname{rad} X.$$

Proof. Let $d = \operatorname{diam} X$ and $r = \operatorname{rad} X$. If $X = \varnothing$, then $r = 0 = d$. Similarly, if X is unbounded, then $r = \infty = d$. Thus, it suffices to consider the case that there is some $a \in X$, and that $X \subseteq \overline{D}(0, M)$ for some $M \geq 0$. In particular, r and d are both finite.

To prove the first inequality, we simply observe that $X \subseteq \overline{D}(a,d)$, and therefore $r \leq d$.

For the second inequality, given any $\varepsilon > 0$, there is some point $b \in X$ and some real number $s \in (r, r + \varepsilon/2)$ such that $X \subseteq \overline{D}(b,s)$. Therefore, for any $x, y \in X$,

$$|x-y|_v = |(x-b)-(y-b)|_v \leq |x-b|_v + |y-b|_v \leq 2s < 2r + \varepsilon. \tag{2.1}$$

Since (2.1) holds for all $\varepsilon > 0$ and all $x, y \in X$, we have $d \leq 2r$. □

Chapter 3

Dynamics

3.1 Filled Julia Sets

The following definition originally appeared in [2, p.98].

Definition 3.1.1. Let $\phi(z) \in \mathbb{Q}(z)$ be a rational function with homogenous presentation

$$\phi([x, y]) = [f(x, y), g(x, y)],$$

where $f, g \in \mathbb{Z}[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$. We say that ϕ has *good reduction* at a prime p if the reductions $\bar{f}$ and $\bar{g}$ modulo p have no common zeros in $\mathbb{F}_p \times \mathbb{F}_p$ besides $(x, y) = (0, 0)$.

Here, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denotes the field with p elements. Naturally, given a homogeneous polynomial $f(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$, the reduction $\bar{f}(x, y)$ in Definition 3.1.1 means $\sum_{i=0}^{d} \bar{a}_i x^i y^{d-i}$.

Good reduction turns out to be closely related to the notion of filled Julia sets. The motivating idea for such sets is that for a polynomial ϕ, all of the interesting dynamics involves points that do not escape to ∞ under iteration.

Definition 3.1.2. Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|\cdot|_v$, and let $\phi(z) \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$. The *filled Julia set* of ϕ at v is

$$\mathcal{K}_v = \big\{x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n \geq 1} \text{ is bounded}\big\}.$$

Example 3.1.3. Fix a prime p, an integer $d \geq 2$ with $p \nmid (d-1)$, and $c \in \mathbb{C}_p$ with $|c|_p > 1$. Set $r = |c|_p$ and $\phi(z) = z^d - c^{d-1}z$. Note that for any $x \in \mathbb{C}_p$ with $|x|_p > r$, we have $|\phi(x)|_p = |x|_p^d$, so that $\phi^n(x) \to \infty$. That is, $\mathcal{K}_p \subseteq \overline{D}(0, r)$; in particular, $\operatorname{rad} \mathcal{K}_p \leq r$.

Lemma 3.1.4. *Let p be a prime number, and let $\phi(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathbb{Q}[z]$ be a polynomial of degree $d \geq 2$. Denote by $\mathcal{K}_p$ the filled Julia set of ϕ in $\mathbb{C}_p$, and set $r = |a_d|_p^{1/(d-1)} \operatorname{rad} \mathcal{K}_p$. If $r > 1$ and $\mathcal{K}_p \cap \mathbb{Q} \neq \varnothing$, then*

$$r \geq \begin{cases} p & \text{if } d = 2, \\ p^{1/[(d-1)(d-2)]} & \text{if } d \geq 3. \end{cases}$$

Proof. Given $b \in \mathcal{K}_p \cap \mathbb{Q}$, we may replace ϕ by $\phi(z+b) - b \in \mathbb{Q}[z]$, which is a polynomial of the same degree and lead coefficient as ϕ, but with filled Julia set translated by $-b$. In particular, the radius r is preserved; thus, we may assume without loss that $0 \in \mathcal{K}_p$.

Choose $\alpha \in \mathbb{C}_p$ such that $\alpha^{d-1} = a_d$, and let j be the largest index between 0 and $d-1$ that maximizes $\lambda_j := |\alpha^{1-j}a_j|_p^{1/(d-j)}$.

Claim 3.1.5. $\lambda_j > 1$.

Proof of Claim 3.1.5. If $\lambda_j \leq 1$, then $|a_i\alpha^{-i}|_p \leq |\alpha|_p^{-1} = |a_d\alpha^{-d}|_p$ for every $i = 0, \ldots, d-1$. Thus, for any $x \in \mathbb{C}_p$ with $|x|_p > |\alpha|_p^{-1}$, we have $|\phi(x)|_p = |a_d x^d|_p$. It follows by induction that

$$|\phi^n(x)|_p = |a_d|_p^{(d^n-1)/(d-1)}|x|_p^{d^n} = |\alpha|_p^{-1}|\alpha x|_p^{d^n} \to \infty$$

as $n \to \infty$, and hence $x \notin \mathcal{K}_p$. Thus, as in Example 3.1.3, $\mathcal{K}_p \subseteq \overline{D}(0, |\alpha|_p^{-1})$, contradicting the hypothesis that $r > 1$. $\square$

By Theorem 6.5.7 of [1], which considers the so-called Newton polygon of ϕ, there is some $\beta \in \mathbb{C}_p$ with $\phi(\beta) = 0$ and $|\alpha\beta|_p = \lambda_j$. We have $0, \alpha\beta \in \mathcal{K}_p$; hence, $r \geq \lambda_j$.

If $j = 0$, then a simple induction shows that $|\alpha\phi^n(0)|_p = |\alpha a_0|_p^{d^{n-1}}$ for $n \geq 1$. However, that contradicts the hypothesis that $0 \in \mathcal{K}_p$, since $|\alpha a_0|_p > 1$.

Thus, $1 \leq j \leq d-1$. Writing $|a_d|_p = p^{e_1}$ and $|a_j|_p = p^{e_2}$, where $e_1, e_2 \in \mathbb{Z}$ and $e_2 \geq 1$, we have

$$r \geq \lambda_j = |\alpha^{1-j}a_j|_p^{1/(d-j)} = p^f > 1, \qquad \text{where} f = \frac{1}{d-j}\left(\frac{(1-j)}{(d-1)}e_1 + e_2\right) > 0.$$

If $j = 1$, then $f = e_2/(d-1) \geq 1/(d-1)$, which proves the Lemma for the case $d = 2$ and part of the case $d \geq 3$. Finally, if $2 \leq j \leq d-1$, then because $f > 0$, we must have $e_1 > (1-d)e_2$. However, e_1 and e_2 are integers, and therefore $e_1 \geq 1 + (d-1)e_2$. Thus, $f \geq 1/[(d-1)(d-j)] \geq 1/[(d-1)(d-2)]$. $\square$

Remark 3.1.6. The bounds of Lemma 3.1.4 are sharp. Indeed, one can check that they are attained by $\phi(z) = z^2 - z/p$ for $d = 2$ and by $\phi(z) = p^d z^d - pz^2$ for $d \geq 3$.

3.2 Elementary Computations

We will write $\log_d x$ to denote the logarithm of x to base d.

Definition 3.2.1. Let $N \geq 0$ and $d \geq 2$ be integers. We define $E(N, d)$ to be twice the sum of all base-d coefficients of all integers from 0 to $N - 1$. That is,

$$E(N,d) = 2\sum_{j=0}^{N-1} e(j,d), \qquad \text{where} \qquad e\left(\sum_{i=0}^{M} c_i d^i, d\right) = \sum_{i=0}^{M} c_i,$$

for $c_i \in \{0, 1, \ldots, d-1\}$.

We will need the following Lemma.

Lemma 3.2.2. *Let $d \geq 2$ and $N \geq 1$ be integers, and write $N = c + dk$ with $0 \leq c \leq d-1$ and $k \geq 0$. Then:*

a. $E(N,d) = (d-c)E(k,d) + cE(k+1,d) + (d-1)N - c(d-c)$.

b. *If $N \leq d$, then $E(N,d) = N(N-1)$.*

c. $(d-c)\log_d\left(\frac{dk}{N}\right) + c\log_d\left(\frac{dk+d}{N}\right) \leq 0$.

d. *If $N \geq d$, then $(d-1)\log_d\left(\frac{dk+d}{N}\right) - (d-c) \leq 0$.*

Proof. Writing an arbitrary integer $j \geq 0$ as $j = i + d\ell$ for $0 \leq i \leq d-1$, we compute

$$\begin{aligned}
E(N,d) &= 2\sum_{j=0}^{N-1} e(j,d) = 2\sum_{i=0}^{c-1}\sum_{\ell=0}^{k} e(i+d\ell,d) + 2\sum_{i=c}^{d-1}\sum_{\ell=0}^{k-1} e(i+d\ell,d) \\
&= 2\sum_{i=0}^{c-1}\sum_{\ell=0}^{k}(i+e(\ell,d)) + 2\sum_{i=c}^{d-1}\sum_{\ell=0}^{k-1}(i+e(\ell,d)) \\
&= \sum_{i=0}^{c-1}[2(k+1)i + E(k+1,d)] + \sum_{i=c}^{d-1}[2ki + E(k,d)] \\
&= cE(k+1,d) + (d-c)E(k,d) + (k+1)c(c-1) + kd(d-1) - kc(c-1).
\end{aligned}$$

Part (a) now follows by rewriting the last three terms as

$$c(c-1) + dk(d-1) = c(c-d) + (c+dk)(d-1) = (d-1)N - c(d-c).$$

For part (b), we simply observe that if $1 \leq N \leq d$, then

$$E(N,d) = 2\big(1 + \cdots + (N-1)\big) = N(N-1).$$

To prove part (c), note that the function $\log_d(x)$ is of course concave down. Letting $x_1 = dk/N$ and $x_2 = (dk+d)/N$, then, we have $x_1 \leq 1 < x_2$, and therefore $\log_d(1) \geq L(1)$, where

$$L(x) = \frac{1}{x_2 - x_1}\left[(x_2 - x)\log_d(x_1) + (x - x_1)\log_d(x_2)\right]$$

is the line through $(x_1, \log_d(x_1))$ and $(x_2, \log_d(x_2))$. That is,

$$0 \geq \frac{1}{d}\left[(d-c)\log_d\left(\frac{dk}{N}\right) + c\log_d\left(\frac{dk+d}{N}\right)\right].$$

For part (d), we have

$$(d-1)\log_d\left(\frac{dk+d}{N}\right) = \frac{(d-1)}{\log d}\cdot\log\left(1 + \frac{d-c}{N}\right) \leq \frac{(d-1)}{\log d}\cdot\frac{(d-c)}{N}.$$

However, $\log d = -\log[1 - (d-1)/d] \geq (d-1)/d$, and since $N \geq d$,

$$(d-1)\log_d\left(\frac{dk+d}{N}\right) \leq (d-1)\cdot\frac{d}{d-1}\cdot\frac{d-c}{N} = \frac{d}{N}(d-c) \leq (d-c). \qquad \square$$

Theorem 3.2.3. *Let $d \geq 2$ and $N \geq 1$ be integers. Then $E(N,d) \leq (d-1)N\log_d N$, with equality if N is a power of d.*

Proof. The result is immediate for $N = 1$ and $N = d$ by Lemma 3.2.2.b. If $1 < N < d$, then because $(\log x)/(x-1)$ is a decreasing function, we have $(\log d)/(d-1) \leq (\log N)/(N-1)$, from which the desired inequality follows.

For $N \geq d+1$, we proceed by induction on N, assuming the result holds for all positive integers up to $N-1$. Write $N = c + dk$, where $0 \leq c \leq d-1$, so that $1 \leq k \leq N-2$. By Lemma 3.2.2.a, we have

$$\begin{aligned} E(N,d) &= (d-c)E(k,d) + cE(k+1,d) + (d-1)N - c(d-c) \\ &\leq (d-c)(d-1)k\log_d k + c(d-1)(k+1)\log_d(k+1) + (d-1)N - c(d-c) \\ &= (d-c)(d-1)k\log_d(dk) + c(d-1)(k+1)\log_d(dk+d) - c(d-c), \end{aligned}$$

where the final equality is because $N = (d-c)k + c(k+1)$, and the inequality (which is equality if N is a power of d) is by the inductive hypothesis, since $k, k+1 \leq N-1$. Adding and subtracting $(d-1)N\log_d N$, then,

$$\begin{aligned} E(N,d) \leq (d-1)N\log_d N &+ (d-c)(d-1)k\log_d\left(\frac{dk}{N}\right) \\ &+ c(d-1)(k+1)\log_d\left(\frac{dk+d}{N}\right) - c(d-c) \\ = (d-1)N\log_d N &+ c\left[(d-1)\log_d\left(\frac{dk+d}{N}\right) - (d-c)\right]. \\ &+ (d-1)k\left[(d-c)\log_d\left(\frac{dk}{N}\right) + c\log_d\left(\frac{dk+d}{N}\right)\right] \end{aligned}$$

The quantities in square brackets are nonpositive by Lemma 3.2.2.c–d, and the Theorem follows. □

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Corrections

When originally submitted, this honors thesis contained some errors which have been corrected in the current version. Here is a list of the errors that were corrected.

Various places in the thesis. Approximately 20 spelling errors were corrected, 10 missing periods or commas were added in mathematical formulae, and approximately 30 spacing and sizing changes were made to mathematical formulae.

Other changes:

p. 1, l. 7. The length of the period of y was changed from n to $n - m$ in two places.

p. 1, l. –6. The reference to [4, Theorem 3] was added.

p. 3. In the first paragraph of the Proof of Proposition 2.1.2, "and hence $|1|_v = 1$" was changed to "and hence either $|1|_v = 0$ or… as desired."

p. 4, l. 9. The phrase "is not complete" was changed to "need not be complete."

p. 5, l. 4. The formula "$ab \in \mathfrak{m}_v$" was changed to "$ba = ab \in \mathfrak{m}_v$".

p. 5, l. 6. Two appearances of "thus" were changed to "hence" and "Therefore".

p. 5, l. 10. The clause "if $X \neq \varnothing$, or 0 if $X = \varnothing$" was added.

p. 5. The sentence "Note that if X …" was added after Definition 2.2.2.

p. 5. The four sentences, "If $X = \varnothing$, then… are both finite" were added to the first paragraph of the Proof of Proposition 2.2.3.

p. 7, l. 1–3. The subscript v was changed to to p in four places.

p. 7. The clauses "let $r' = \operatorname{rad} \mathcal{K}_p$, and set $r = |a_d|_p^{1/(d-1)} r'$" were changed to "set $r = |a_d|_p^{1/(d-1)} \operatorname{rad} \mathcal{K}_p$."

p. 7, l. 13. The exponent $1/d - j$ was changed to $1/(d-j)$.

p. 8, l. 8. The sentence, "We will write $\log_d x \dots$" was added.

p. 9, l. 2–4. On each of these three lines, $\sum_{i=c-1}^{d-1}$ was changed to to $\sum_{i=c}^{d-1}$.

p. 9, l. –4. In the first inequality of this line, $\leq$ was changed to to $\geq$.

p. 10, l. –2. The word "negative" was changed to to "nonpositive".